An Implicit Smooth Reformulation of Complementarity Constraints for Application to Robust Structural Optimization

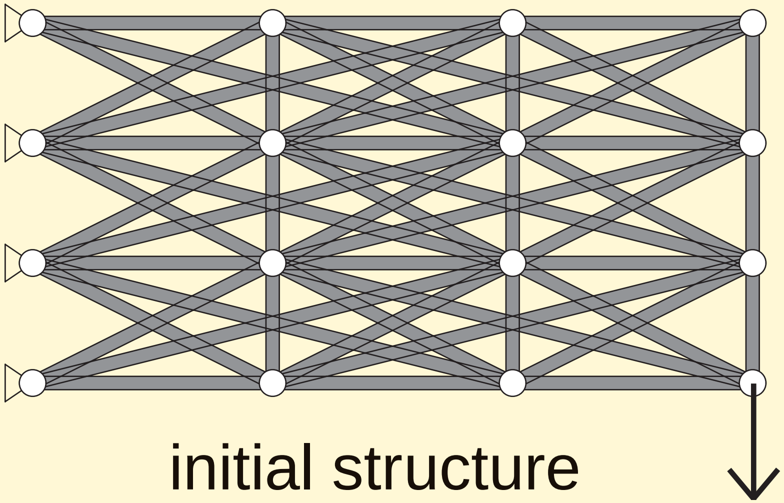
Yoshihiro Kanno

University of Tokyo (Japan)

September 7, 2010

Motivation: Robust Structural Optimization

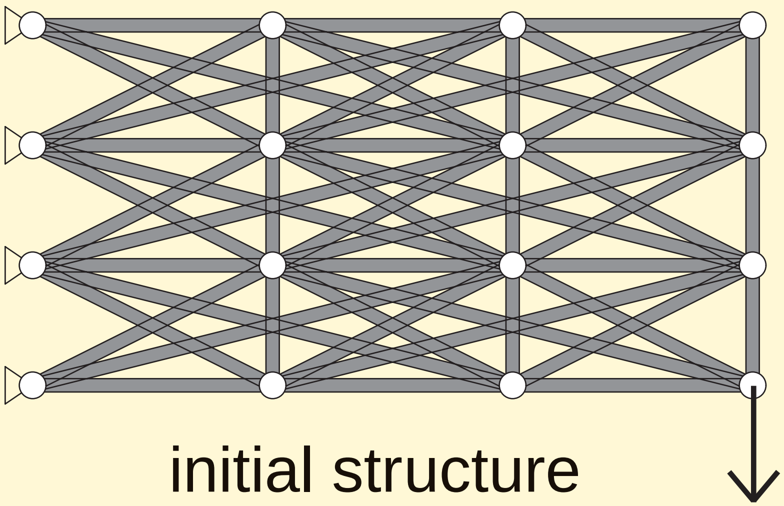
- structural optimization
  - constraints — $g_i(\boldsymbol{x}) \geq 0$
  -
  -



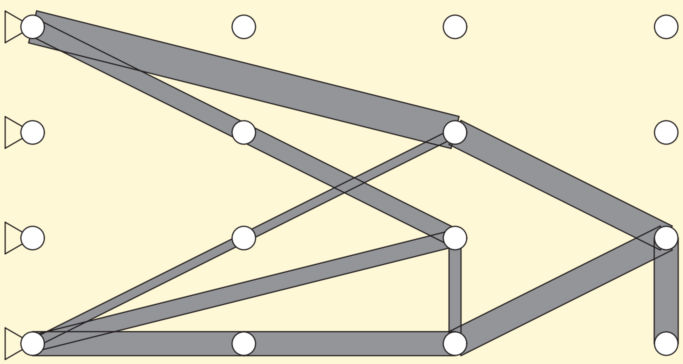
initial structure

Motivation: Robust Structural Optimization

- structural optimization
  - constraints — $g_i(\boldsymbol{x}) \geq 0$
  - uncertainty in loads, manufacturing variability, aging,...
  - $\rightarrow$ robust optimization



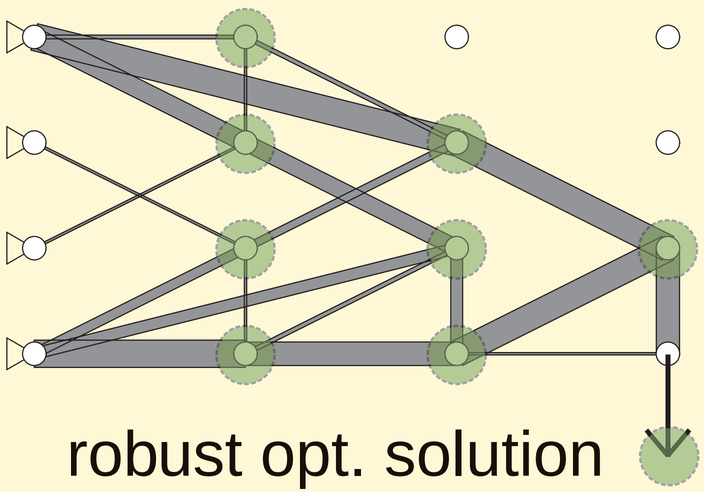
initial structure



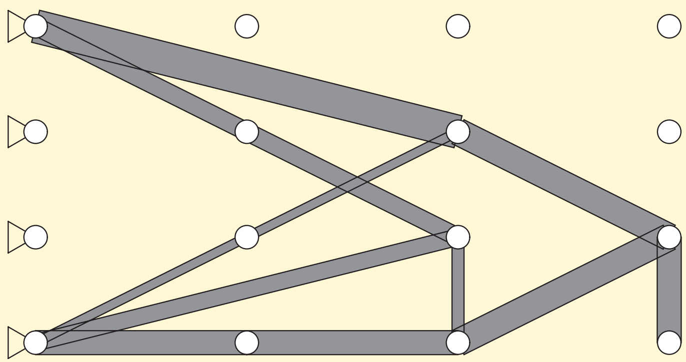
conventional opt. solution

Motivation: Robust Structural Optimization

- structural optimization
  - constraints — $g_i(\boldsymbol{x}) \geq 0$
  - uncertain loads
  - robust constraint satisfaction — $g_i(\boldsymbol{x}; \boldsymbol{\zeta}) \geq 0 \ (\forall \boldsymbol{\zeta} \in \mathcal{Z})$



robust opt. solution



conventional opt. solution

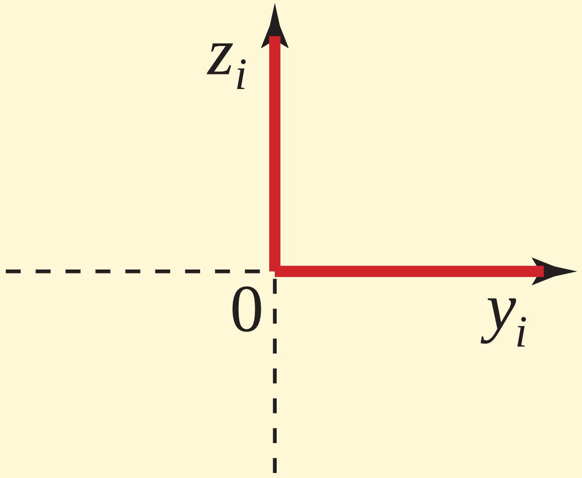
- optimization with complementarity constraints
  $\rightarrow$ propose a reformulation attacked by an NLP approach

MPEC (in Complementarity Form)

$$
\begin{array}{lll}
\min\limits_{\boldsymbol{x}} & f_0(\boldsymbol{x}) & \\
\text{s.t.} & f_j(\boldsymbol{x}) \le 0, & j = 1, \ldots, m \\
 & g_i(\boldsymbol{x}) \ge 0,\ h_i(\boldsymbol{x}) \ge 0,\ g_i(\boldsymbol{x}) h_i(\boldsymbol{x}) = 0, & i = 1, \ldots, n
\end{array}
$$

- MPEC — mathematical program with equilibrium constraints
- complementarity condition

$$
y_i \ge 0, \quad z_i \ge 0, \quad y_i z_i = 0
$$



MPEC (in Complementarity Form)

$$\begin{array}{lll} \min\limits_{\boldsymbol{x}} & f_0(\boldsymbol{x}) & \\ \text{s.t.} & f_j(\boldsymbol{x}) \le 0, & j = 1, \ldots, m \\ & g_i(\boldsymbol{x}) \ge 0,\ h_i(\boldsymbol{x}) \ge 0,\ g_i(\boldsymbol{x}) h_i(\boldsymbol{x}) = 0, & i = 1, \ldots, n \end{array}$$

- MPEC — mathematical program with equilibrium constraints
- complementarity condition

$$y_i \ge 0, \quad z_i \ge 0, \quad y_i z_i = 0$$

- does not satisfy any constraint qualification
- $\rightarrow$ standard NLP (nonlinear programming) approach
  cannot be applied

Existing Methods of Robust Structural Optimization

- probabilistic approach
- possibilistic approach
  - convex model [Ben-Haim & Elishakoff 90]
    - [Elishakoff, Haftka & Fang 94] [Ganzerli & Pantelides 98], [Au, Cheng, Tham & Zheng 03] [Jiang, Han & Liu 07]
  - 1st-order approximation
    - [Lee & Park 01]
  - semidefinite program
    - compliance [Ben-Tal & Nemirovski 97]
    - stress [Kanno & Takewaki 06]

Robust Optimization (Possibilistic Approach)

- nominal (conventional) structural optimization

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathcal{X}} \quad & vol(\boldsymbol{x}) \\ \text{s.t.} \quad & g_i(\boldsymbol{u}) \geq 0, \quad \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f} \end{aligned}$$

  - constraint on the mechanical performance

$$g_i(\boldsymbol{u}) = b_i - \boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{u} \geq 0$$

  - $\boldsymbol{x}$: cross-sectional areas $\boldsymbol{K}(\boldsymbol{x})$: stiffness matrix
    $\boldsymbol{u}$: displacement $\boldsymbol{f}$: external load

Robust Optimization (Possibilistic Approach)

- nominal (conventional) structural optimization

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathcal{X}} \quad & vol(\boldsymbol{x}) \\ \text{s.t.} \quad & g_i(\boldsymbol{u}) \geq 0, \quad \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f} \end{aligned}$$

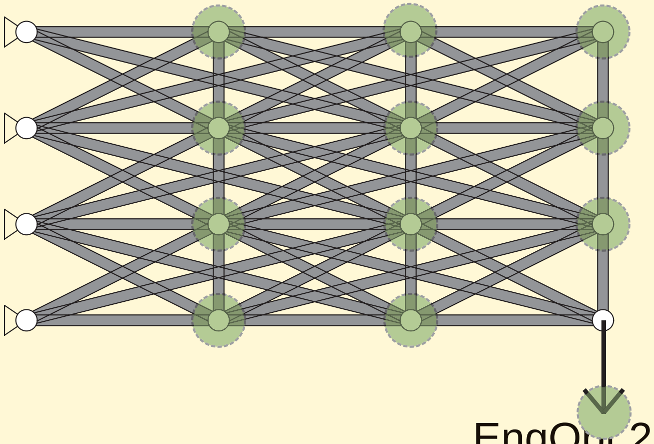
  - constraint on the mechanical performance

$$g_i(\boldsymbol{u}) = b_i - \boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{u} \geq 0$$

- robust constraint

$$\max_{\boldsymbol{u}} \{ g_i(\boldsymbol{u}) \mid \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F} \} \geq 0 \qquad (♠)$$

- $\mathcal{F}$: uncertainty set of external loads



Robust Optimization (Possibilistic Approach)

- nominal (conventional) structural optimization

$$\min_{\boldsymbol{x} \in \mathcal{X}} \quad vol(\boldsymbol{x})$$
$$\text{s.t.} \quad g_i(\boldsymbol{u}) \geq 0, \quad \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}$$

  - constraint on the mechanical performance

$$g_i(\boldsymbol{u}) = b_i - \boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{u} \geq 0$$

- robust constraint

$$\max_{\boldsymbol{u}} \{ g_i(\boldsymbol{u}) \mid \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F} \} \geq 0 \qquad (\spadesuit)$$

- $\mathcal{F}$: uncertainty set of external loads

- robust optimization

$$\min_{\boldsymbol{x} \in \mathcal{X}} \quad vol(\boldsymbol{x})$$
$$\text{s.t.} \quad (\spadesuit)$$

Uncertainty Model

- nominal load : $\tilde{\boldsymbol{f}} \in \mathbf{R}^n$
- uncertainty set

$$\mathcal{F} = \{\tilde{\boldsymbol{f}} + \boldsymbol{F}_0 \boldsymbol{\zeta} \mid \alpha \geq \|\boldsymbol{\zeta}\|_\infty\}$$

  - $\boldsymbol{F}_0$: constant matrix
  - $\alpha \geq 0$: level of uncertainty

Uncertainty Model

- nominal load : $\tilde{\boldsymbol{f}} \in \mathbf{R}^n$
- uncertainty set

$$\mathcal{F} = \{\tilde{\boldsymbol{f}} + \boldsymbol{F}_0 \boldsymbol{\zeta} \mid \alpha \geq \|\boldsymbol{\zeta}\|_\infty\}$$

- robust constraint

$$\max_{\boldsymbol{u}} \{\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{u} \mid \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}\} \leq b_i \qquad (\spadesuit)$$

can be rewritten in terms of the KKT condition

MPEC Formulation

- robust optimization

$$\begin{array}{ll} \min\limits_{\boldsymbol{x}\in\mathcal{X}} & vol(\boldsymbol{x}) \\ \text{s.t.} & (\spadesuit) \end{array}$$

- MPEC formulation

$$\begin{array}{ll} \min & v(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X}, \\ & \forall i = 1, \dots, n : \\ & \quad \left\{ \begin{array}{ll} \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u}_j - \tilde{\boldsymbol{f}} - \boldsymbol{F}_0\boldsymbol{\zeta}_i = \boldsymbol{0}, & \\ \boldsymbol{a}_i + \boldsymbol{K}(\boldsymbol{x})\boldsymbol{\mu}_i = \boldsymbol{0}, & \\ \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{u}_i \le b_i, & \\ \boldsymbol{F}_0^{\mathrm{T}}\boldsymbol{\mu}_i + \boldsymbol{\lambda}_i - \boldsymbol{\tau}_i = \boldsymbol{0}, & \\ \alpha - \zeta_{pi} \ge 0,\ \lambda_{pi} \ge 0,\ \lambda_{pi}(\alpha - \zeta_{pi}) = 0, & \forall p, \\ \alpha + \zeta_{pi} \ge 0,\ \tau_{pi} \ge 0,\ \tau_{pi}(\alpha + \zeta_{pi}) = 0, & \forall p. \end{array} \right. \end{array}$$

Existing Methods for MPEC

- specialized algorithms for MPEC
- reformulation of MPEC

$\leftarrow$ attacked by a standard NLP algorithm

$$g_i(\boldsymbol{x}) \geq 0, \quad h_i(\boldsymbol{x}) \geq 0, \quad g_i(\boldsymbol{x})h_i(\boldsymbol{x}) = 0 \qquad (\spadesuit)$$

Existing Methods for MPEC

- specialized algorithms for MPEC
- reformulation of MPEC

$\leftarrow$ attacked by a standard NLP algorithm

$$g_i(\boldsymbol{x}) \geq 0, \quad h_i(\boldsymbol{x}) \geq 0, \quad g_i(\boldsymbol{x})h_i(\boldsymbol{x}) = 0 \qquad (\spadesuit)$$

- regularization [Scholtes 01] $(\rho \searrow 0)$

$$g_i(\boldsymbol{x}) \geq 0, \quad h_i(\boldsymbol{x}) \geq 0, \quad g_i(\boldsymbol{x})h_i(\boldsymbol{x}) \leq \rho$$

Existing Methods for MPEC

- specialized algorithms for MPEC
- reformulation of MPEC

  $\leftarrow$ attacked by a standard NLP algorithm

$$g_i(\boldsymbol{x}) \geq 0, \quad h_i(\boldsymbol{x}) \geq 0, \quad g_i(\boldsymbol{x})h_i(\boldsymbol{x}) = 0 \qquad (\spadesuit)$$

  - regularization [Scholtes 01] $\quad (\rho \searrow 0)$

$$g_i(\boldsymbol{x}) \geq 0, \quad h_i(\boldsymbol{x}) \geq 0, \quad g_i(\boldsymbol{x})h_i(\boldsymbol{x}) \leq \rho$$

  - reformulation (nonsmooth)

$$\phi(g_i(\boldsymbol{x}), h_i(\boldsymbol{x}); 0) = 0 \quad \Leftrightarrow \quad (\spadesuit)$$

  - smoothing [Fukushima, Luo & Pang 98] [Tin-Loi 99] $\quad (\rho \searrow 0)$

$$\phi(g_i(\boldsymbol{x}), h_i(\boldsymbol{x}); \rho) = 0$$

Reformulation and Smoothing

- reformulation (nonsmooth)
  - Fischer–Burmeister function:

$$\psi(y, z) = y + z - \sqrt{y^2 + z^2} = 0$$

  - $\min$-function:

$$\psi(y, z) = \min\{y, z\} = 0$$

Reformulation and Smoothing

- reformulation (nonsmooth)
  - Fischer–Burmeister function:

$$\psi(y,z) = y + z - \sqrt{y^2 + z^2} = 0$$

- smoothing ($\leftarrow$ solved by NLP)

$$\phi(y,z,0) = \psi(y,z)$$

$\phi(y,z,\rho)$ is continuously differentiable $(\forall \rho > 0)$

  - smoothed Fischer–Burmeister function:

$$\phi(y,z,\rho) = y + z - \sqrt{y^2 + z^2 + 2\rho^2} = 0$$

  - CHKS smoothing function (Chen–Harker–Kanzow–Smale) :

$$\phi(y,z,\rho) = y + z - \sqrt{(y-z)^2 + 4\rho^2} = 0$$

Implicit Reformulation

- smoothed Fischer–Burmeister function:

$$\phi(y, z, \rho) = y + z - \sqrt{y^2 + z^2 + 2\rho^2} = 0$$

  - ρ: smoothing parameter ($\rho \searrow 0$)
  - choice of an initial value for ρ?
  - decreasing strategy of ρ?

Implicit Reformulation

- smoothed Fischer–Burmeister function:

$$\phi(y, z, \rho) = y + z - \sqrt{y^2 + z^2 + 2\rho^2} = 0$$

  - ρ: smoothing parameter ($\rho \searrow 0$)
  - choice of an initial value for ρ?
  - decreasing strategy of ρ?

- idea:
  - regard ρ as a variable
  - add a constraint so that $\rho = 0$ at the optimal solution
  - adjust ρ to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z}$ (the residual of CC)

Implicit Reformulation

- smoothed Fischer–Burmeister function

$$\phi(y_i, z_i, \rho) = y_i + z_i - \sqrt{y_i^2 + z_i^2 + 2\rho^2} = 0 \quad (i = 1, \ldots, n) \quad (\diamondsuit)$$

- additional constraint

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = n(e^{\rho} - 1) \quad (\clubsuit)$$

Implicit Reformulation

- smoothed Fischer–Burmeister function

$$\phi(y_i, z_i, \rho) = y_i + z_i - \sqrt{y_i^2 + z_i^2 + 2\rho^2} = 0 \quad (i = 1, \ldots, n) \quad (\diamondsuit)$$

- additional constraint

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = n(e^{\rho} - 1) \qquad (\clubsuit)$$

- $(\diamondsuit)$ & $(\clubsuit)$ $\Leftrightarrow$ $y_i \geq 0,\ z_i \geq 0,\ y_i z_i = 0\ (\forall n)$, $\rho = 0$
- ρ is regarded as a variable

Implicit Reformulation

- smoothed Fischer–Burmeister function

$$\phi(y_i, z_i, \rho) = y_i + z_i - \sqrt{y_i^2 + z_i^2 + 2\rho^2} = 0 \quad (i = 1, \ldots, n) \quad (\diamondsuit)$$

- additional constraint

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = n(e^{\rho} - 1) \qquad (\clubsuit)$$

- $(\diamondsuit)$ & $(\clubsuit)$ $\Leftrightarrow$ $y_i \geq 0,\ z_i \geq 0,\ y_i z_i = 0\ \ (\forall n)$, $\rho = 0$
- ρ is regarded as a variable
- ρ serves as a measure of the residual of CC:

$$(\clubsuit) \quad \Leftrightarrow \quad \log\left(\frac{1}{n}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} + 1\right) = \rho$$

$|\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z}|$ is large $\leftrightarrow$ $|\rho|$ is large $\qquad \boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} \to 0 \leftrightarrow \rho \to 0$

Implicit Reformulation

- MPEC

$$\begin{array}{lll} \min\limits_{\boldsymbol{x}} & f_0(\boldsymbol{x}) & \\ \text{s.t.} & f_j(\boldsymbol{x}) \leq 0, & \forall j \\ & g_i(\boldsymbol{x}) \geq 0,\ h_i(\boldsymbol{x}) \geq 0,\ g_i(\boldsymbol{x})h_i(\boldsymbol{x}) = 0, & \forall i \end{array}$$

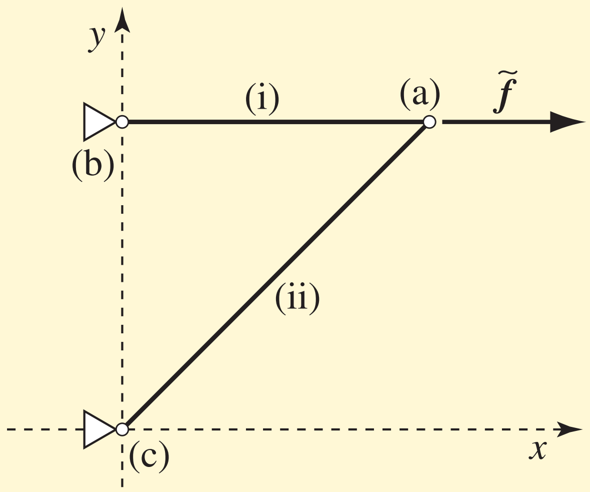
- reformulation

$$\left.\begin{array}{lll} \min\limits_{\boldsymbol{x},\rho} & f_0(\boldsymbol{x}) & \\ \text{s.t.} & f_j(\boldsymbol{x}) \leq 0, & \forall j \\ & \phi(g_i(\boldsymbol{x}), h_i(\boldsymbol{x}), \rho) = 0, & \forall i \\ & \sum_{i=1}^{n} g_i(\boldsymbol{x})h_i(\boldsymbol{x}) = n(e^{\rho} - 1) & \end{array}\right\} \qquad (\heartsuit)$$

- solve (♡) by using a standard NLP

  e.g., SQP, interior-point method

Ex.) 2-bar Truss

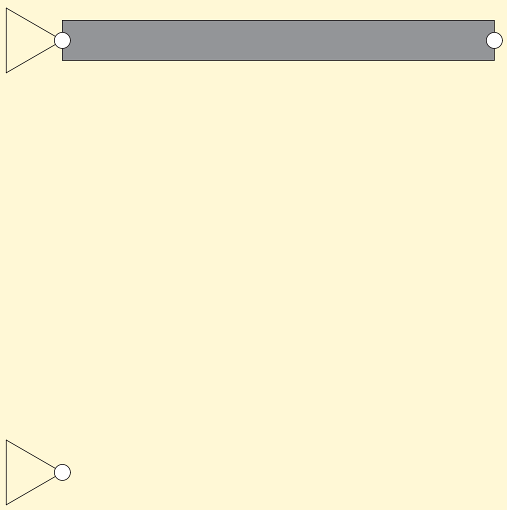


- uncertainty in external load

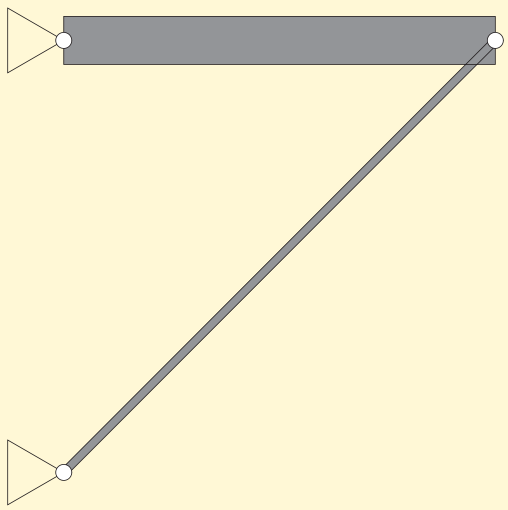
$$\boldsymbol{f} = \tilde{\boldsymbol{f}} + \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix},$$

$$\alpha \geq |\zeta_j| \ (\forall j)$$

- stress constraints



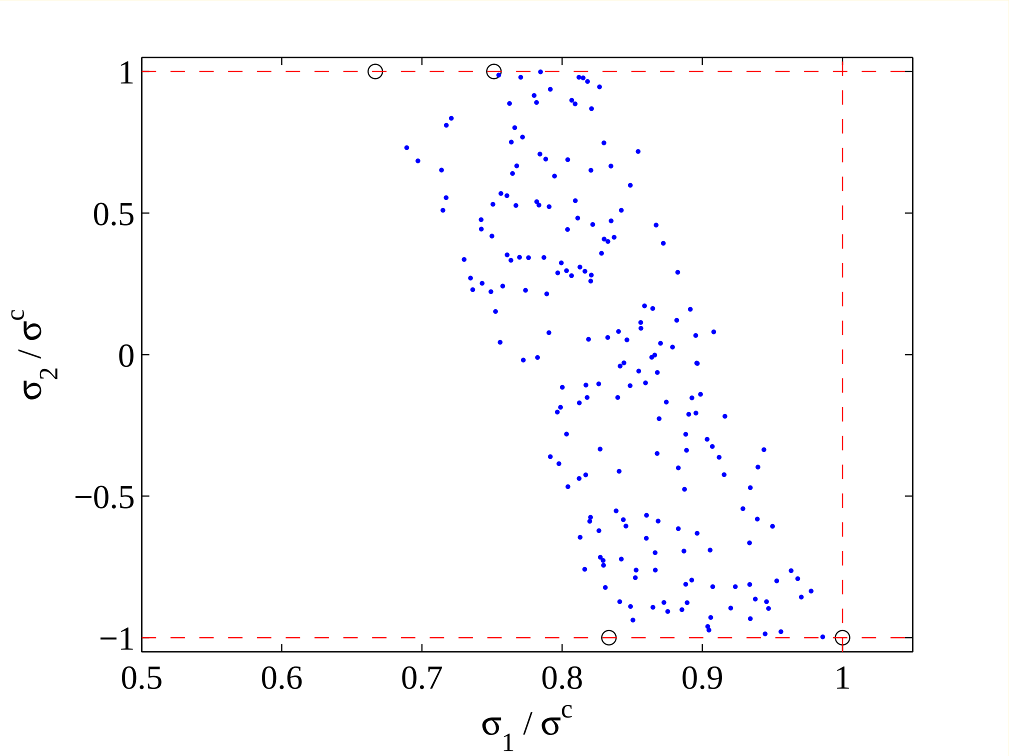
nominal opt.



robust opt.

Ex.) 2-bar Truss

- worst-case scenarios



stress states of robust opt. solution

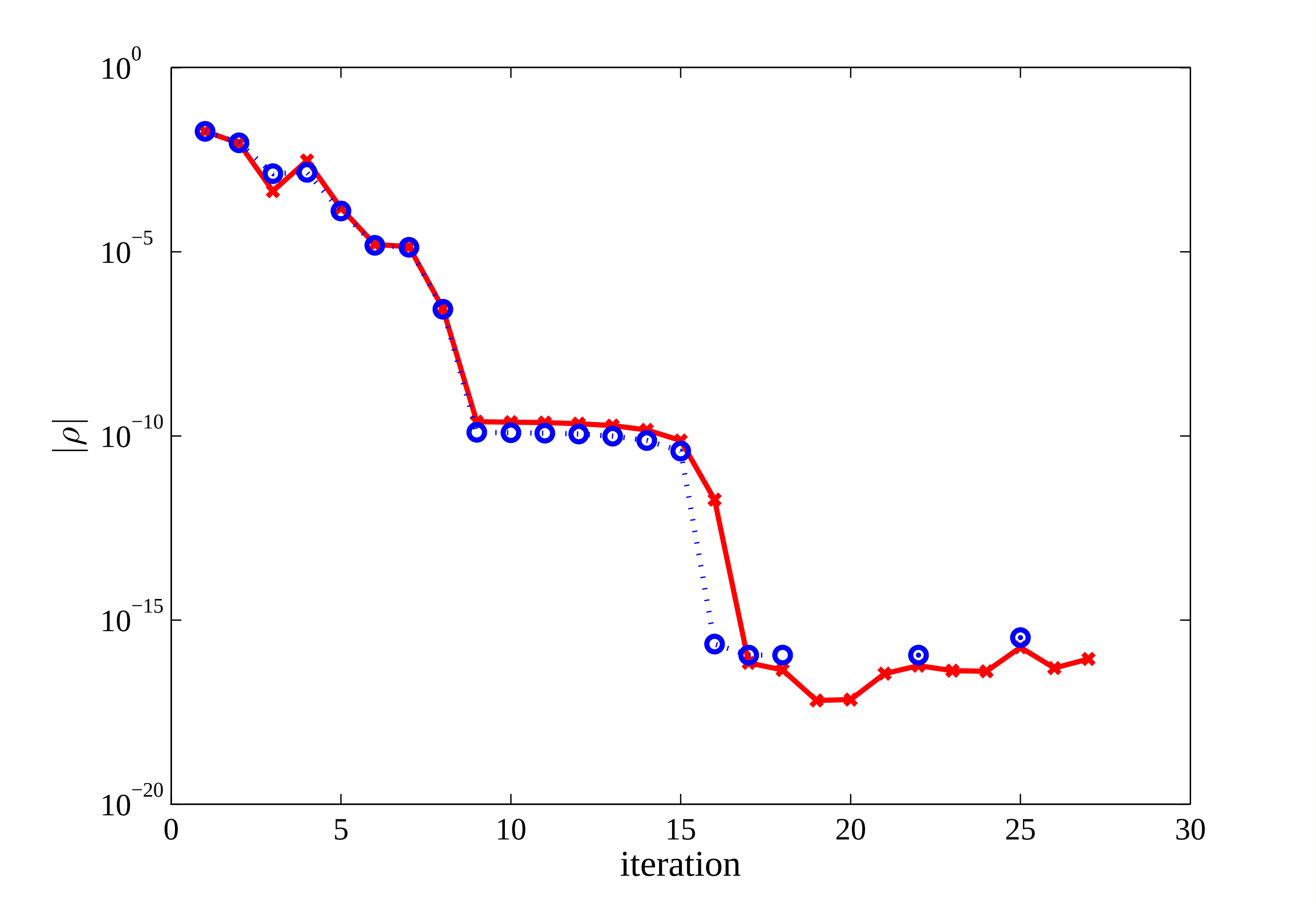
- each stress constraint become active in the worst case

Ex.) 2-bar Truss

- the smoothing parameter ρ

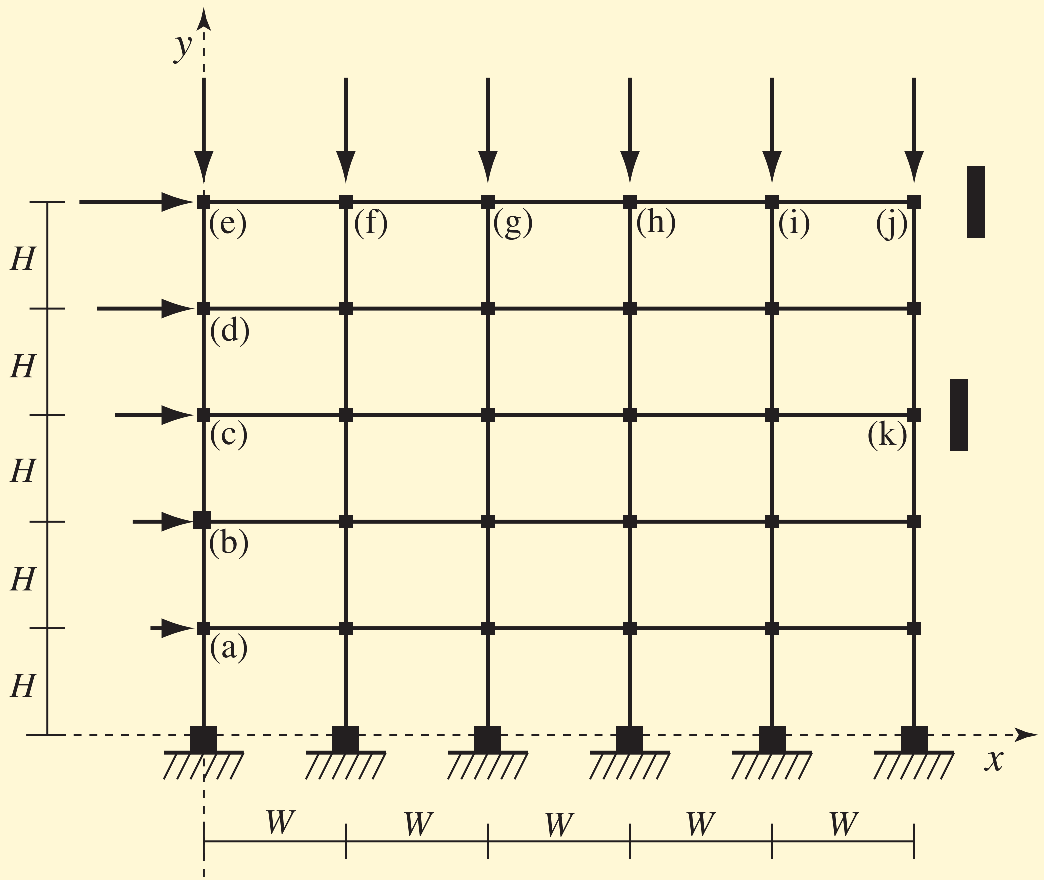
  vs

  the residual of the complementarity constraints $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = 0$



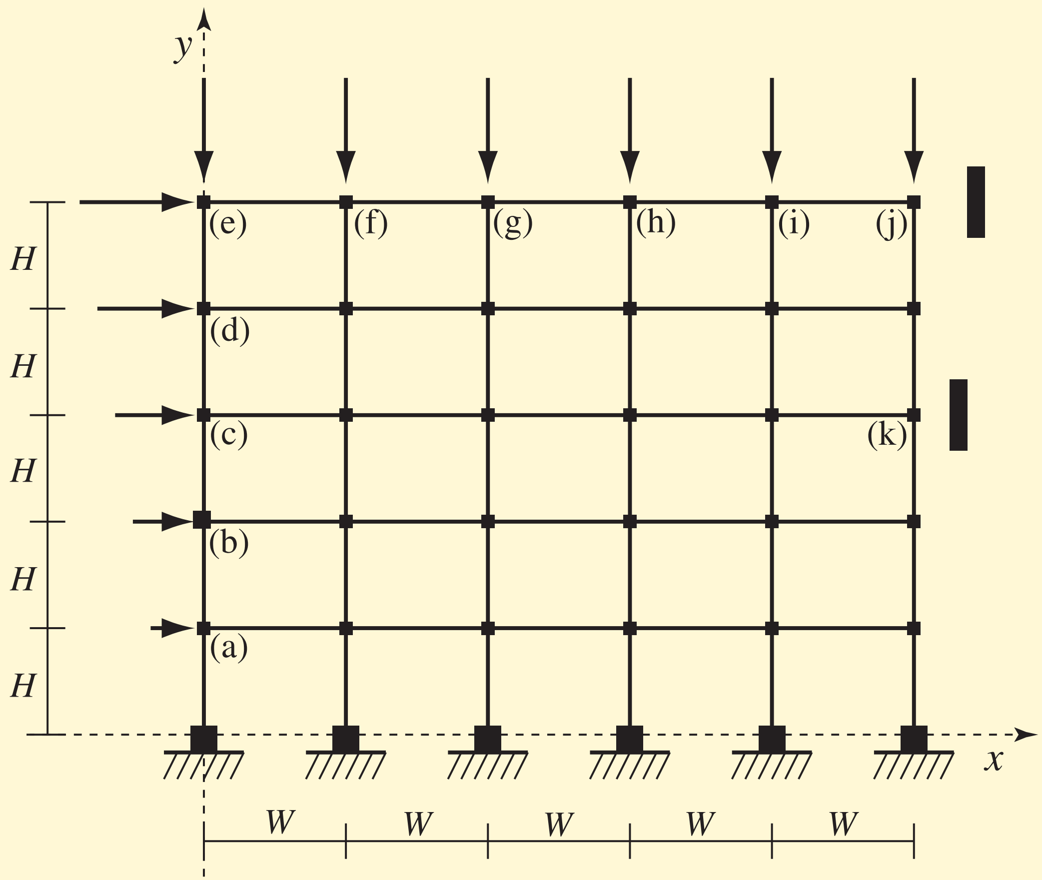
$$\rho = \log\left(\frac{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z}}{n} + 1\right)$$

Ex.) 5-story Frame



- plane frame
- sandwich cross-section
  — $t_i = r_i^2 x_i$
  - t_i : 2nd moment of inertia
  - x_i : cross-sectional area
  - $2r_i$: distance between flanges

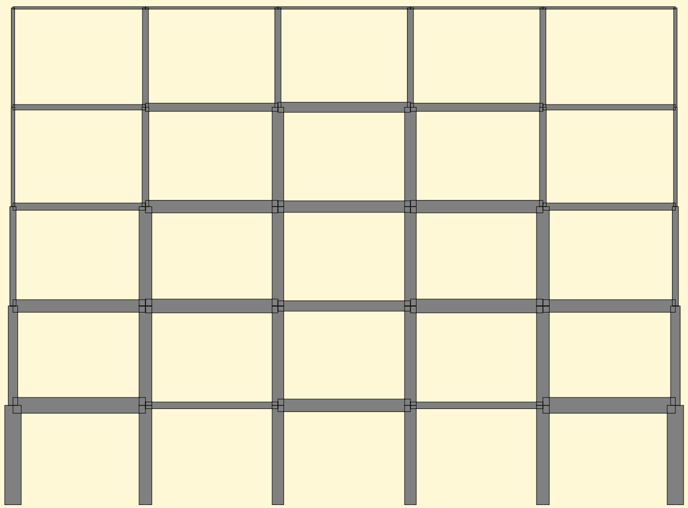
Ex.) 5-story Frame



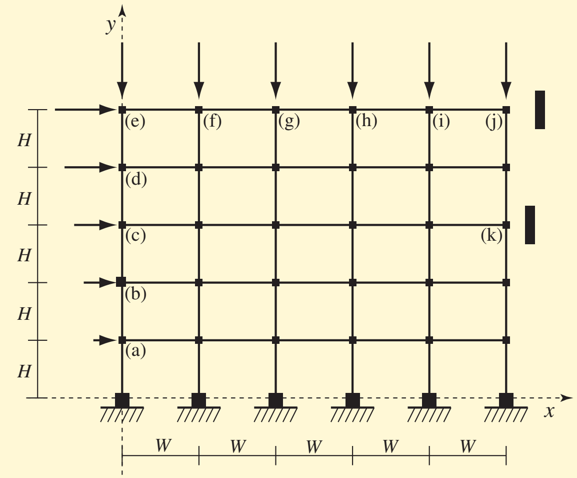
- plane frame
- sandwich cross-section
  — $t_i = r_i^2 x_i$
  - t_i : 2nd moment of inertia
  - x_i : cross-sectional area
  - $2r_i$: distance between flanges

- displacement constraints
- 776 variables, 240 complementarity constraints

Ex.) 5-story Frame

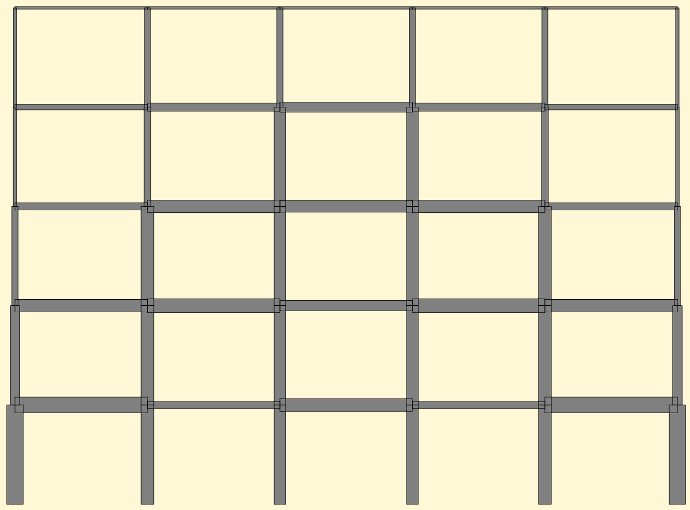


nominal opt.



model

Ex.) 5-story Frame

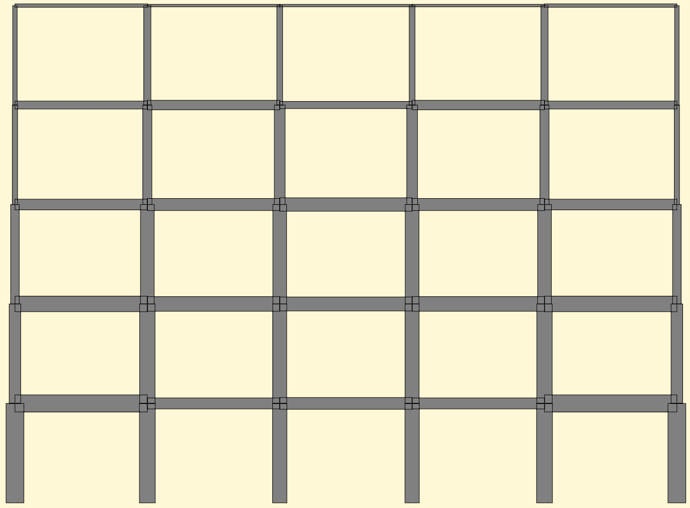


nominal opt.

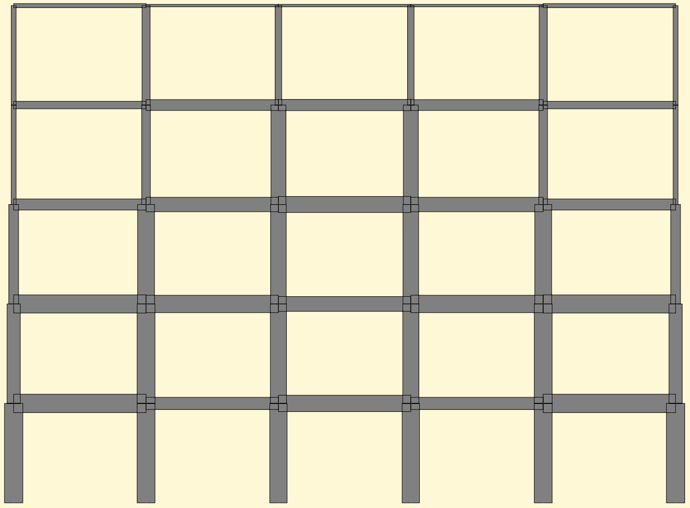
- uncertainty in external load

$$\boldsymbol{f} = \tilde{\boldsymbol{f}} + \boldsymbol{F}_0 \boldsymbol{\zeta}, \quad \alpha \ge \|\boldsymbol{\zeta}\|_\infty$$

- α : level of uncertainty



$\alpha = 0.1$



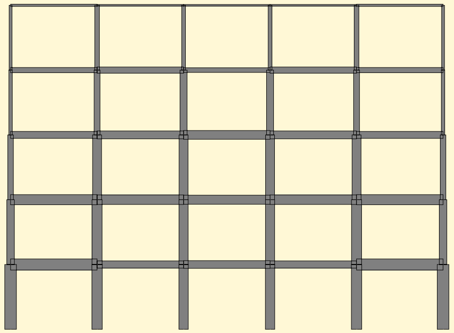
$\alpha = 0.2$

Ex.) 5-story Frame (Computational Result)

- interior-point method: `fmincon` (Malab)

α	Volume (m^3)	CPU (sec)	Iter.
nominal	14.1948	13.4	66
0.05	15.3665	238.1	122
0.10	16.5896	118.8	57
0.15	17.7392	283.1	144
0.20	18.9299	245.6	125

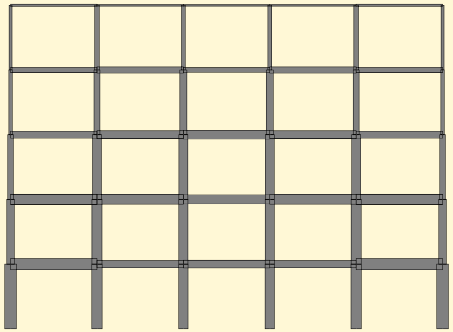
Ex.) 5-story Frame



$\alpha = 0.1$

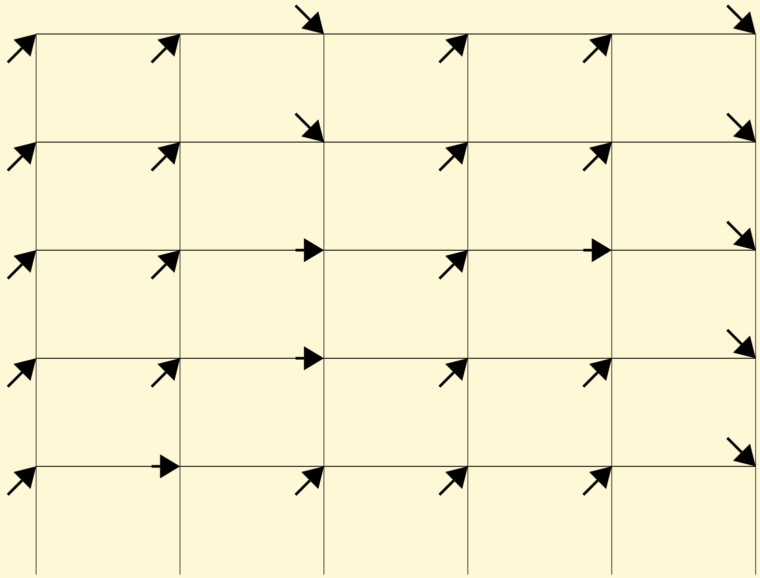
- worst-case scenarios

Ex.) 5-story Frame

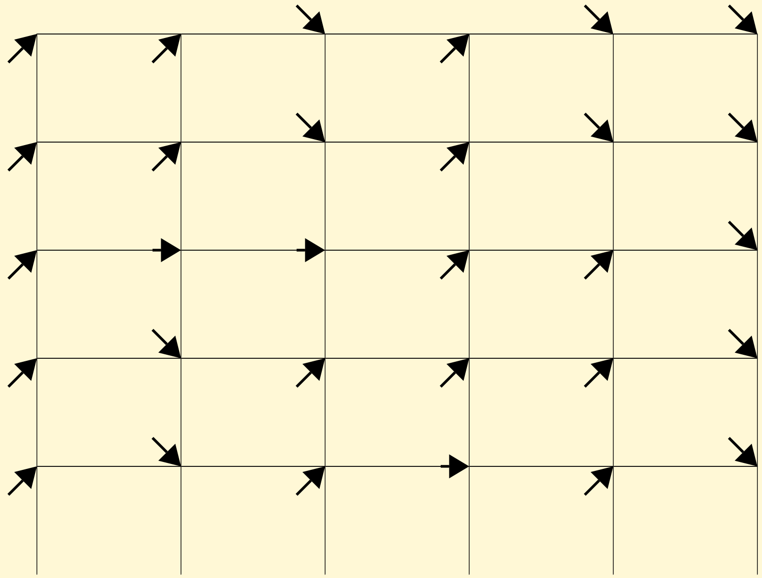


$\alpha = 0.1$

- worst-case scenarios

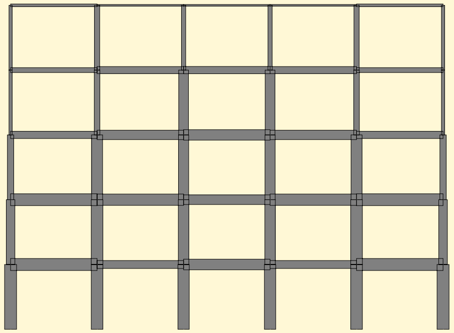


for displacement (j)



for displacement (k)

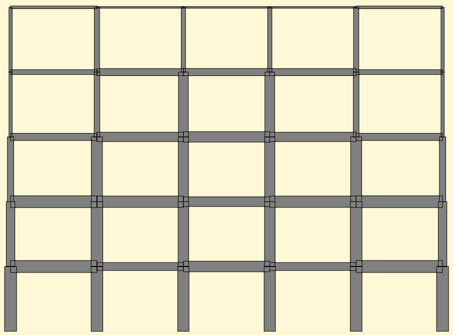
Ex.) 5-story Frame



$\alpha = 0.2$

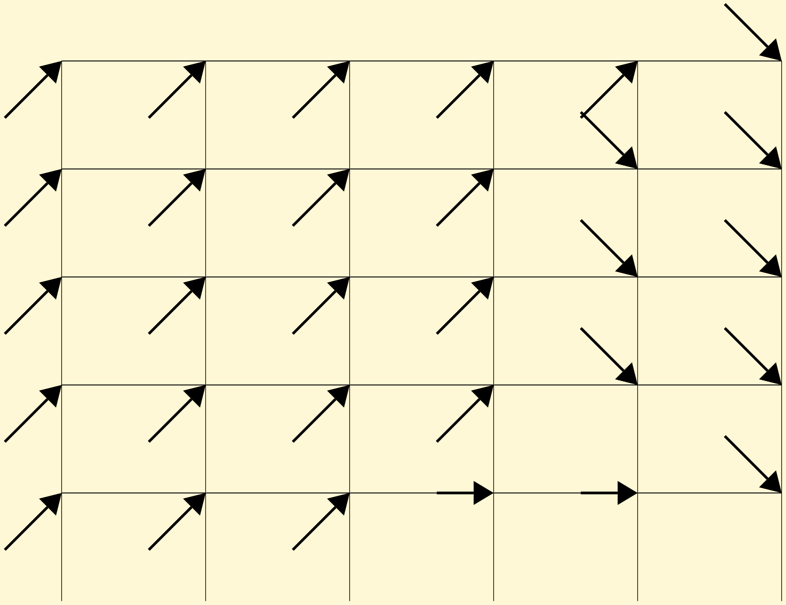
- worst-case scenarios

Ex.) 5-story Frame

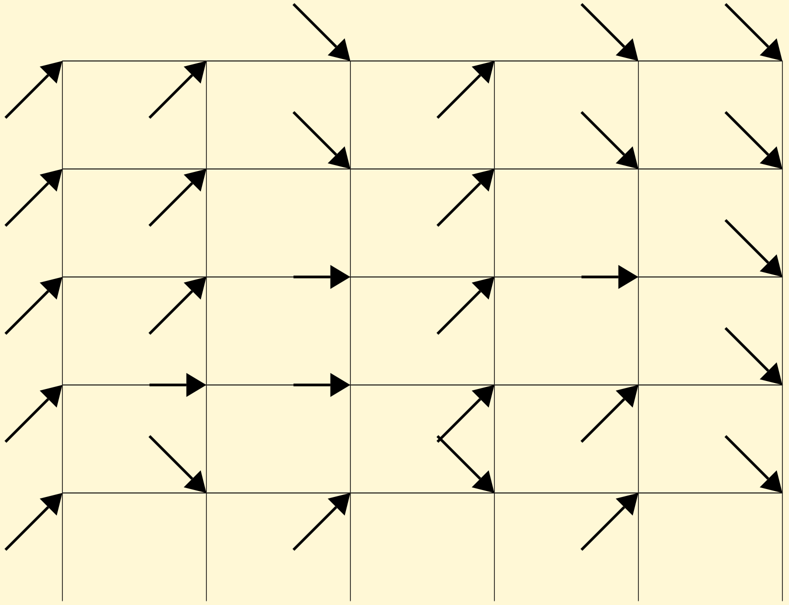


$\alpha = 0.2$

- worst-case scenarios

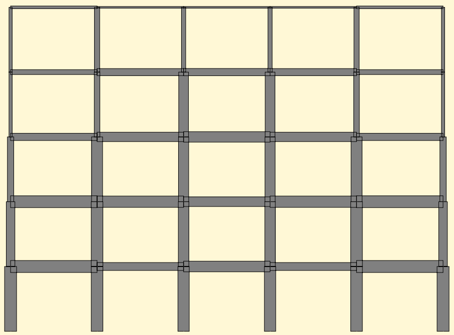


for displacement (j)



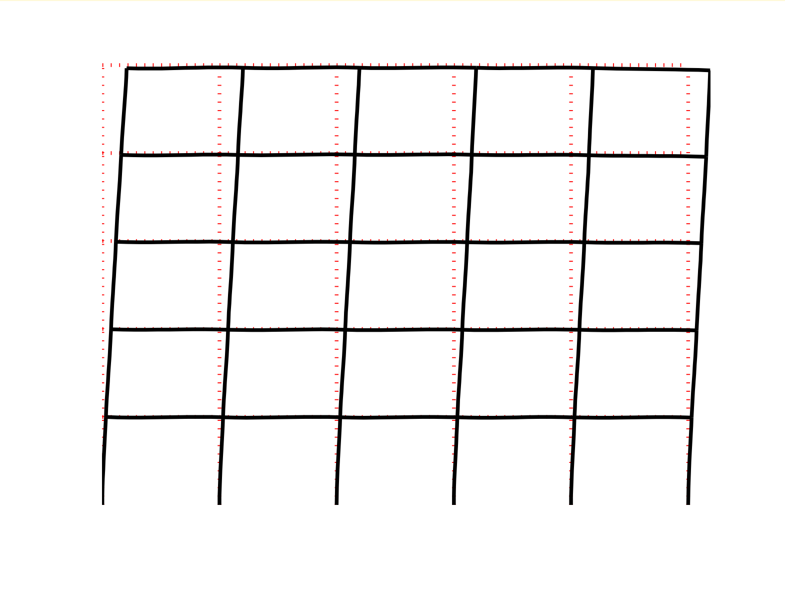
for displacement (k)

Ex.) 5-story Frame

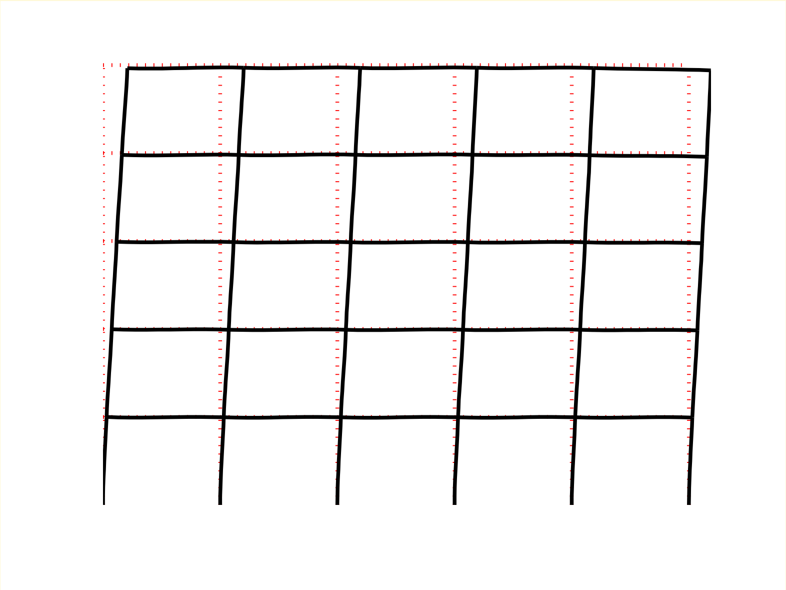


$\alpha = 0.2$

- worst-case scenarios — responses



for displacement (j)



for displacement (k)

Conclusions

- robust structural optimization
  - uncertainty in external load
  - MPEC — involving complementarity constraints
- reformulation and smoothing
  - complementarity function
  - smoothing with a parameter ρ
    Fischer–Burmeister/CHKS functions
- implicit reformulation
  - treat ρ as an independent variable
  - ρ is related to the residual of CC
  - solved by a standard NLP approach