Robust Truss Topology Optimization
with Discrete Design Variables
via Mixed Integer Programming

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Motivation: Robust Topology Optimization

- truss topology optimization
  - under single load $\rightarrow$ opt. solution is often unstable
  - $\rightarrow$ robust optimization is required

![Initial Truss vs Conventional Opt. Solution](image_url)
Motivation: Robust Topology Optimization

• truss topology optimization [Ben-Tal & Nemirovski 97]
  • uncertain loads at all nodes, in all directions
  • minimize the compliance in the worst-case

initial truss

robust opt. solution
Motivation: Robust Topology Optimization

- truss topology optimization
  - uncertain loads at all nodes, in all directions
  - minimize the compliance in the worst-case

- stable solution is always obtained
- all nodes remain \(\Rightarrow\) topology is not (necessarily) optimal
Motivation: Robust Topology Optimization

- truss topology optimization
  - uncertain loads at all nodes, in all directions
  - minimize the compliance in the worst-case

→ propose a robust truss optimization formulation considering the variation of topology
Existing Methods of Robust Truss Optimization

- probabilistic approach
- possibilistic approach
  - convex model [Ben-Haim & Elishakoff 90]
    - [Elishakoff, Haftka & Fang 94] [Ganzerli & Pantelides 98], [Au, Cheng, Tham & Zheng 03] [Jiang, Han & Liu 07]
  - 1st-order approximation
    - [Lee & Park 01]
  - semidefinite program
    - compliance [Ben-Tal & Nemirovski 97] all nodes remain
    - stress [Kanno & Takewaki 06] stress constraints are not removed

\[ x_i \geq \epsilon \]
Robust Optimization (Possibilistic Approach)

• nominal (conventional) truss optimization

\[
\begin{align*}
\min_{x \in X} & \quad vol(x) \\
\text{s.t.} & \quad g_q(u) \leq 0, \quad K(x)u = f
\end{align*}
\]

• \( g_q(u) \leq 0 \) : constraint on the mechanical performance

• \( x \) : cross-sectional areas, \( K(x) \) : stiffness matrix,
  \( u \) : displacement, \( f \) : external load
Robust Optimization (Possibilistic Approach)

• nominal (conventional) truss optimization

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\begin{align*}
\min_{x \in \mathcal{X}} & \quad vol(x) \\
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\end{align*}
\]

• \(g_q(u) \leq 0\) : constraint on the mechanical performance

• robust constraint

\[
\max_u \{g_q(u) \mid K(x)u \in \bar{\mathcal{F}}\} \leq 0
\]

• \(\bar{\mathcal{F}}\) : uncertainty set of external loads

Robust Truss Topology Optimization via MIP
Robust Optimization (Possibilistic Approach)

- nominal (conventional) truss optimization

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\min_{x \in \mathcal{X}} & \quad \text{vol}(x) \\
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- \( \bar{F} \) : uncertainty set of external loads

- robust optimization

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\min_{x \in \mathcal{X}} & \quad \text{vol}(x) \\
\text{s.t.} & \quad (\spadesuit)
\end{align*}
\]
Uncertainty Model

- nominal load: \( \tilde{f} \in \mathbb{R}^n \)

- uncertainty set

\[
\tilde{\mathcal{F}} = \{ \tilde{f} + F_0 \zeta \mid \alpha \geq \| \zeta_j \| \ (\forall j) \}
\]
Uncertainty Model

- nominal load: \( \tilde{f} \in \mathbb{R}^n \)

- uncertainty set

\[
\bar{F} = \{ \tilde{f} + F_0 \zeta \mid \alpha \geq \Vert \zeta_j \Vert (\forall j) \}
\]

- \( F_0 \): constant matrix
- \( \alpha \geq 0 \): level of uncertainty
- \( j \): index of node
- uncertainty at all nodes
Uncertainty Model

- nominal load: \( \tilde{f} \in \mathbb{R}^n \)

- uncertainty set
  \[
  \tilde{F} = \{ \tilde{f} + F_0 \zeta \mid \alpha \geq \| \zeta_j \| \ (\forall j) \}
  \]

- topology-dependent uncertainty model
  \[
  F(s) = \{ \tilde{f} + F_0 \zeta \mid \alpha s_j \geq \| \zeta_j \| \ (\forall j) \}
  \]
**Uncertainty Model**

- **nominal load**: \( \tilde{f} \in \mathbb{R}^n \)

- **uncertainty set**

\[
\tilde{F} = \{ \tilde{f} + F_0 \zeta \mid \alpha \geq \| \zeta_j \| \ (\forall j) \}
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- **topology-dependent uncertainty model**

\[
F(s) = \{ \tilde{f} + F_0 \zeta \mid \alpha s_j \geq \| \zeta_j \| \ (\forall j) \}
\]

- **\( s_j \)**

\[
s_j = \begin{cases} 
1 & \text{if the } j\text{th node exists} \\
0 & \text{if the } j\text{th node is removed}
\end{cases}
\]
Worst-Case Detection

- worst case of response $g_q$

$$\max_u \{ g_q(u) \mid K(x)u \in \mathcal{F}(s) \}$$

- $\mathcal{F}(s)$: uncertainty set of external loads
Worst-Case Detection

- worst case of response $g_q$

$$\max_u \{g_q(u) \mid K(x)u \in F(s)\}$$

- $F(s)$: uncertainty set of external loads

- counter-example: $\max_u u_y$?

$x_3 = \varepsilon$: (♠) provides w.c.  $x_3 = 0$: w.c. is infeasible for (♠)
Worst-Case Detection

- worst case of response $g_q$

\[
\max_u \left\{ g_q(u) \mid K(x)u \in \mathcal{F}(s) \right\}
\]  

- $\mathcal{F}(s)$: uncertainty set of external loads

- a truss is stable $\Rightarrow$ (♣) provides w.c.

- otherwise not necessarily

- stability constraint is necessary for robust truss topology optimization
Discrete Design Variables

- **member cross-sectional area** $x_i$
  
  $$x_i \in \{0, \xi_1, \ldots, \xi_k\}$$

- **0–1 variables** $t_{ip}$
  
  $$x_i = \sum_{p=1}^{k} \xi_p t_{ip}, \quad \sum_{p=1}^{k} t_{ip} \leq 1$$

  e.g., [Stolpe & Svanberg 03]
Discrete Design Variables

- member cross-sectional area $x_i$
  
  $$x_i \in \{0, \xi_1, \ldots, \xi_k\}$$

- 0–1 variables $t_{ip}$
  
  $$x_i = \sum_{p=1}^{k} \xi_p t_{ip}, \quad \sum_{p=1}^{k} t_{ip} \leq 1$$

- indices of area $t_{ip}$ & index of node
  
  $$t_{ip} \leq s_j \leq 1$$

  $$s_j \leq \sum_{i \in \mathcal{I}_j} \sum_{p=1}^{k} t_{ip}$$

  - $i \in \mathcal{I}_j \iff$ member $i$ is connected to node $j$
Stress Constraints

- worst case of stress $\sigma_i$

$$\max_u \{ \sigma_i(u) \mid K(x)u \in \mathcal{F}(s) \} \leq \bar{\sigma} \quad (\diamond a)$$

$$\min_u \{ \sigma_i(u) \mid K(x)u \in \mathcal{F}(s) \} \geq -\bar{\sigma} \quad (\diamond b)$$

- constraints (◊) are rewritten by using the KKT conditions (with several Lagrange multipliers)
Stress Constraints

- worst case of stress $\sigma_i$

\[
\max_u \{ \sigma_i(u) \mid K(x)u \in F(s) \} \leq \bar{\sigma} \quad (\Diamond a)
\]
\[
\min_u \{ \sigma_i(u) \mid K(x)u \in F(s) \} \geq -\bar{\sigma} \quad (\Diamond b)
\]

- constraints ($\Diamond$) are rewritten by using the KKT conditions (with several Lagrange multipliers)

- $x_i = 0 \Rightarrow$ stress constraint should be removed

\[
|\sigma_i(u)| \leq \bar{\sigma} + M(1 - t_i), \quad t_i = \begin{cases} 
1 & \text{if } x_i > 0 \\
0 & \text{if } x_i = 0
\end{cases}
\]

e.g., [Stolpe & Svanberg 03]

- constraints on the associated Lagrange multipliers should also be removed
Global Optimization

- MIP (mixed integer programming) formulation
  - discrete cross-sectional areas
  - topology-dependent uncertainty model
    - uncertainty loads exist only at existing nodes
  - stress constraint in the worst case
    - imposed only to the existing members
  - stability constraint
    - a necessary condition is considered
Ex.) 26-member truss ($\mathcal{X} = \{0, 20\}^m$, $\| \tilde{f} \| = 10.0$)

Nominal opt.

Initial truss
Ex.) 26-member truss \( \mathcal{X} = \{0, 20\}^m, \|\tilde{f}\| = 10.0 \)

- Robust optimal topology depends on the level of uncertainty.
Ex.) 26-member truss \((\mathcal{X} = \{0, 20\}^m, \|\tilde{f}\| = 10.0)\)

nominal opt.

robust opt. \((\alpha = 1.0)\)

robust opt. \((\alpha = 1.5)\)

robust opt. \((\alpha = 3.0)\)
Ex.) 26-member truss (computational result)

- MIP solver: CPLEX Ver. 11.2

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Vol. (cm$^3$)</th>
<th>CPU (s)</th>
<th># of Nodes</th>
</tr>
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<tbody>
<tr>
<td>nominal</td>
<td>9656.9</td>
<td>$\leq$ 0.1</td>
<td>24</td>
</tr>
<tr>
<td>1.0</td>
<td>13656.9</td>
<td>4029.8</td>
<td>13348</td>
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<tr>
<td>1.5</td>
<td>14485.3</td>
<td>9241.6</td>
<td>92693</td>
</tr>
<tr>
<td>3.0</td>
<td>16485.3</td>
<td>630338.8</td>
<td>4231298</td>
</tr>
<tr>
<td>3.3</td>
<td>16485.3</td>
<td>29507.6</td>
<td>133683</td>
</tr>
</tbody>
</table>
Ex.) 26-member truss \((\mathcal{X} = \{0, 20\}^m, \| \tilde{f} \| = 7.5)\)

nominal opt.

initial truss
Ex.) 26-member truss ($\mathcal{X} = \{0, 20\}^m$, $\|\tilde{f}\| = 7.5$)

Robust Truss Topology Optimization via MIP 

nominal opt.

initial truss

robust opt. ($\alpha = 1.0$)

robust opt. ($\alpha = 2.0$)
Ex.) 29-member truss \((\alpha = 1.0)\)

- robust optimal topology depends on the set of candidates of cross-sectional areas
- robust optimal topology is not necessarily unique
Ex.) 29-member truss \( (\alpha = 1.0) \)

- robust optimal topology depends on the set of candidates of cross-sectional areas
- robust optimal topology is not necessarily unique
Ex.) 29-member truss \((\alpha = 1.0)\)

Robust opt.
\((\mathcal{X} = \{0, 10\}^m)\)
Ex.) 29-member truss ($\alpha = 1.0$)

Robust opt. ($\mathcal{X} = \{0, 10\}^m$)

Robust opt. ($\mathcal{X} = \{0, 5, 10\}^m$)
Ex.) 29-member truss ($\alpha = 1.0$)

robust opt. ($X = \{0, 10\}^m$)

robust opt. ($X = \{0, 5, 10\}^m$)

robust opt. ($X = \{0, 5, 15\}^m$)
Ex.) 29-member truss ($\alpha = 1.0$)

Robust opt. ($\mathcal{X} = \{0, 5, 10\}^m$)

worst-case member stresses
Ex.) 29-member truss ($\alpha = 1.0$)

Robust opt. ($\mathcal{X} = \{0, 5, 15\}^m$)

worst-case member stresses
Conclusions

- robust truss optimization
  - topology optimization
  - topology-dependent uncertainty model
    uncertain loads at all existing nodes
- stress constraints
  - $-\bar{\sigma} \leq \text{(stress in the worst case)} \leq \bar{\sigma}$ for all existing members
  - constraint on stability is required
- global optimization
  - Mixed Integer Programming formulation