Robust Truss Topology Optimization under Geometric Uncertainties via Semidefinite Programming

Daiki Hashimoto, Yoshihiro Kanno
University of Tokyo (Japan)

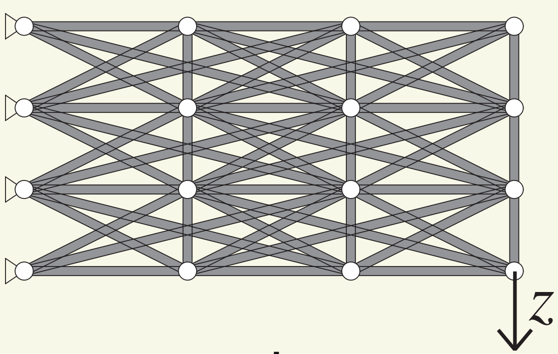
May 27, 2014 (CJK-OSM8)

robust optimization

- nominal (i.e., conventional) compliance optimization

$$\begin{aligned} \min \quad & \pi(\boldsymbol{a}; z) \\ \text{s. t.} \quad & \boldsymbol{a} \geq \boldsymbol{0}, \quad vol(\boldsymbol{a}) \leq \bar{V} \end{aligned}$$

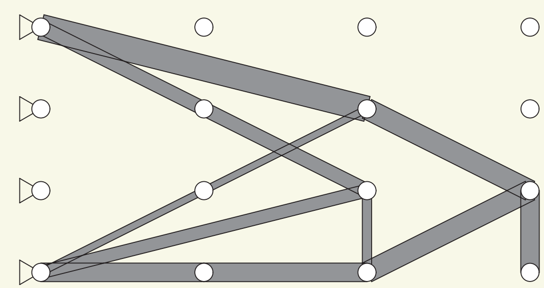
  - $\pi(\boldsymbol{a}; z)$: compliance
  - $\boldsymbol{a}$: cross-sectional areas $\qquad z$: data (e.g., external load)



ground struct.

$\longrightarrow$

optimize



nominal opt. sol.

robust optimization

- nominal (i.e., conventional) compliance optimization

$$\begin{array}{ll} \min & \pi(\boldsymbol{a}; z) \\ \text{s.t.} & \boldsymbol{a} \geq \boldsymbol{0}, \quad vol(\boldsymbol{a}) \leq \bar{V} \end{array}$$

  - $\pi(\boldsymbol{a}; z)$: compliance
  - $\boldsymbol{a}$: cross-sectional areas $\quad z$: data (e.g., external load)

- robust compliance optimization

$$\begin{array}{ll} \min & \max\{\pi(\boldsymbol{a}; z) \mid z \in \mathcal{U}\} \\ \text{s.t.} & \boldsymbol{a} \geq \boldsymbol{0}, \quad vol(\boldsymbol{a}) \leq \bar{V} \end{array}$$

  - $\mathcal{U}$: uncertainty set (e.g., set of uncertain loads)
  - obj. fcn.: worst value of compliance

robust optimization

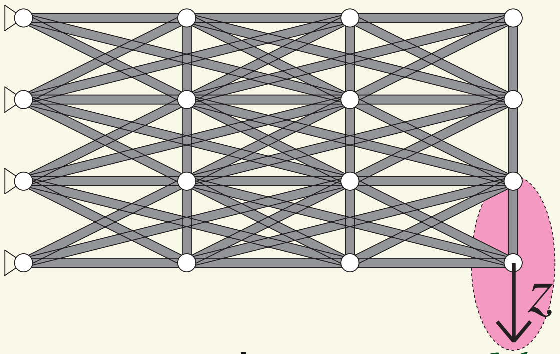
- nominal (i.e., conventional) compliance optimization

$$\begin{array}{ll} \min & \pi(\boldsymbol{a}; z) \\ \text{s.t.} & \boldsymbol{a} \geq \mathbf{0}, \quad vol(\boldsymbol{a}) \leq \bar{V} \end{array}$$

  - $\pi(\boldsymbol{a}; z)$: compliance
  - $\boldsymbol{a}$: cross-sectional areas z : data (e.g., external load)

- robust compliance optimization

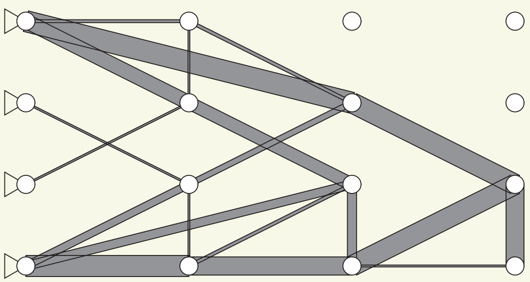
$$\begin{array}{ll} \min & \max\{\pi(\boldsymbol{a}; z) \mid z \in \mathcal{U}\} \\ \text{s.t.} & \boldsymbol{a} \geq \mathbf{0}, \quad vol(\boldsymbol{a}) \leq \bar{V} \end{array}$$



uncertainty: $z \in \mathcal{U}$

$\longrightarrow$

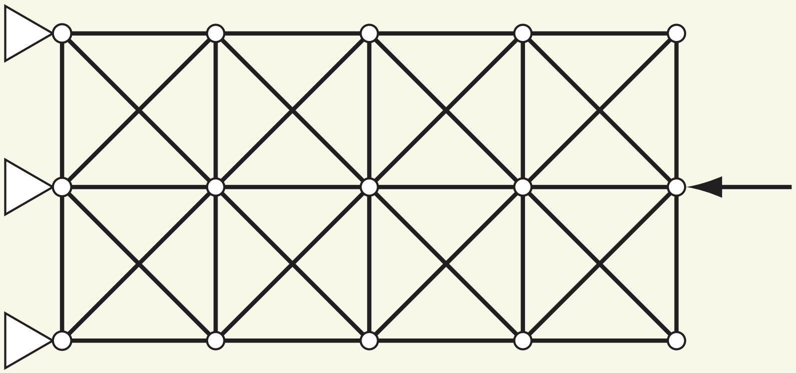
optimize



robust opt. sol.

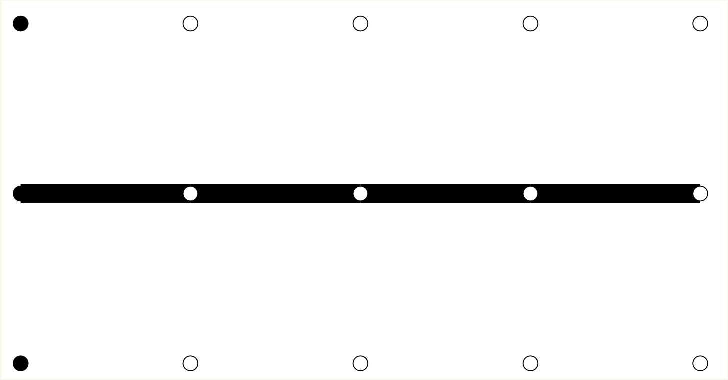
geometric uncertainty: motivation

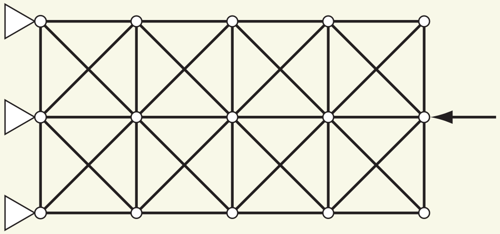
- compliance minimization of a truss
  - problem setting:



geometric uncertainty: motivation

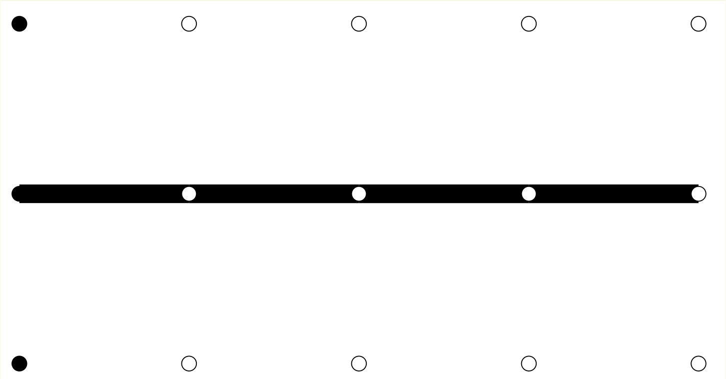
- compliance minimization of a truss
  - optimal solution:



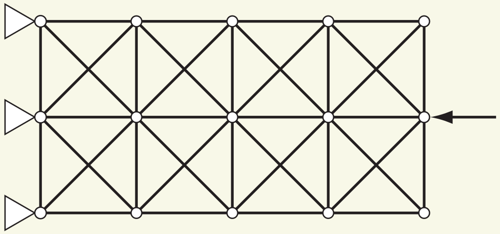


geometric uncertainty: motivation

- compliance minimization of a truss
  - optimal solution:

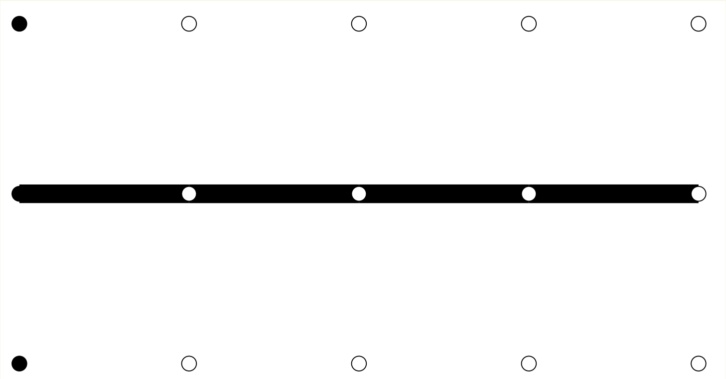


    - unstable (kinematically indeterminate)
    - unrealistic design under compression force
    - imperfection in nodal locations → no equilibrium state

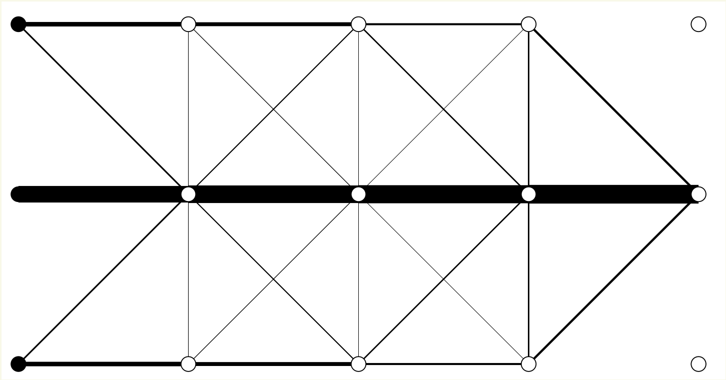


geometric uncertainty: motivation

- compliance minimization of a truss
  - optimal solution:



  - robust optimal solution:



    - robustness against uncertainty in nodal locations

robust truss optimization

- many studies concerning uncertain loads
  - possibilistic (i.e., non-probabilistic) model
    - given set of loads
    - min. the worst compliance [Ben-Tal & Nemirovski 97] [Cherkaev & Cherkaev 03] [Calafiore & Dabbene 08] [de Gournay, Allaire, & Jouve 08] [Takezawa, Nii, Kitamura, & Kogiso 11]

robust truss optimization

- many studies concerning uncertain loads
  - possibilistic (i.e., non-probabilistic) model
    - given set of loads
    - min. the worst compliance [Ben-Tal & Nemirovski 97]

- uncertainty in nodal locations
  - probabilistic model [Guest & Igusa 08]
    - nodal locs.: random variables
    - min. the expected val. of compliance
  - possibilistic model [this study]

uncertainty in boundary shape of continuum

- uniform manufacturing error
  - SIMP [Sigmund 09], [Wang, Lazarov, & Sigmund 11]
  - level-set method [Jang, van Dijk, van Keulen 12]
- probabilistic model
  - SIMP
    [Schevenels, Lazarov, & Sigmund 11] [Lazarov, Schevenels, & Sigmund 12]
  - level-set method [Chen & Chen 11]
- possibilistic model (1st order approx.) [Guo, Zhang, & Zhang 13]

uncertainty in boundary shape of continuum

- uniform manufacturing error
  - SIMP [Sigmund 09], [Wang, Lazarov, & Sigmund 11]
  - level-set method [Jang, van Dijk, van Keulen 12]
- probabilistic model
  - SIMP [Schevenels, Lazarov, & Sigmund 11] [Lazarov, Schevenels, & Sigmund 12]
  - level-set method [Chen & Chen 11]
- possibilistic model (1st order approx.) [Guo, Zhang, & Zhang 13]

- conservative reformulation under large uncertainty
  - semidefinite programming (SDP) [this study]

SDP for robust optimization

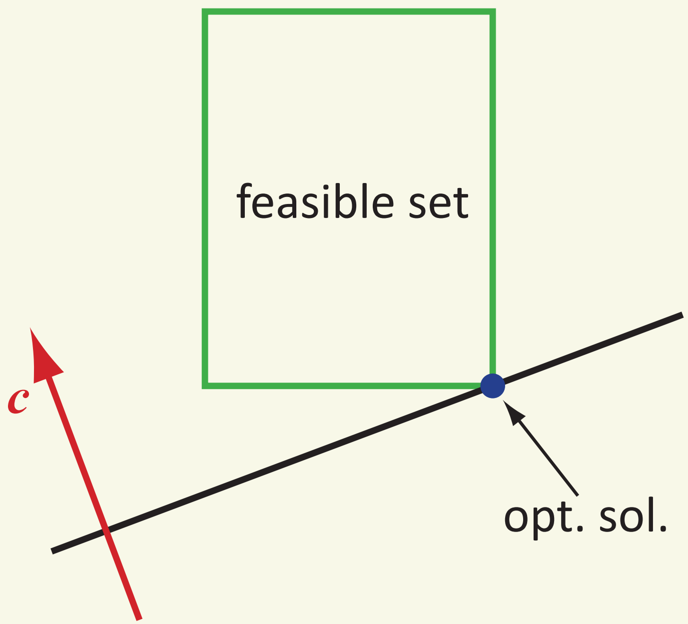
- SDP — convex
  - compliance, load unc. [Ben-Tal & Nemirovski 97]
- nonlinear SDP — nonconvex
  - stress constraints, load unc. [Kanno & Takewaki 06]
  - stiffness unc. (safe approx.) [Guo, Bai, Zhang, & Gao 09]
  - stiffness & length unc. (safe approx.) [Guo, Du, & Gao 11]

SDP for robust optimization

- SDP — convex
  - compliance, load unc. [Ben-Tal & Nemirovski 97]
- nonlinear SDP — nonconvex
  - stress constraints, load unc. [Kanno & Takewaki 06]
  - stiffness unc. (safe approx.) [Guo, Bai, Zhang, & Gao 09]
  - stiffness & length unc. (safe approx.) [Guo, Du, & Gao 11]

- SDP — convex
  - nodal locs. unc. (safe approx.) [this study]

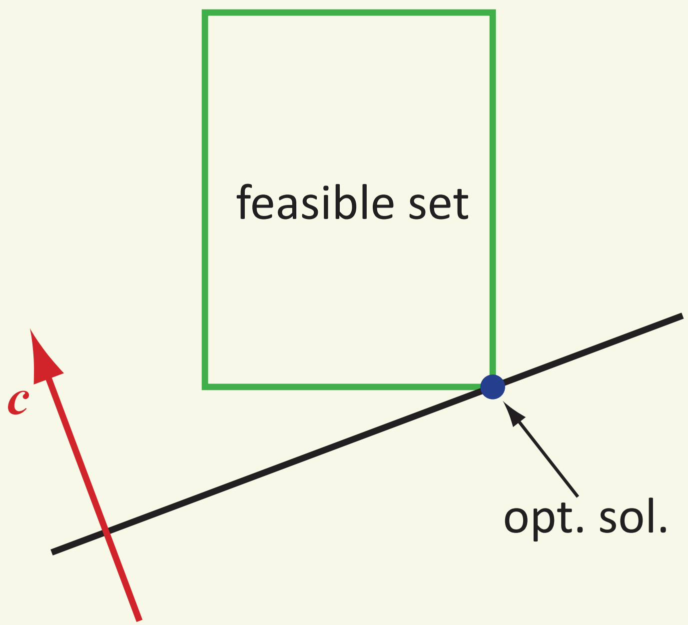
robust opt. & conservative approx.

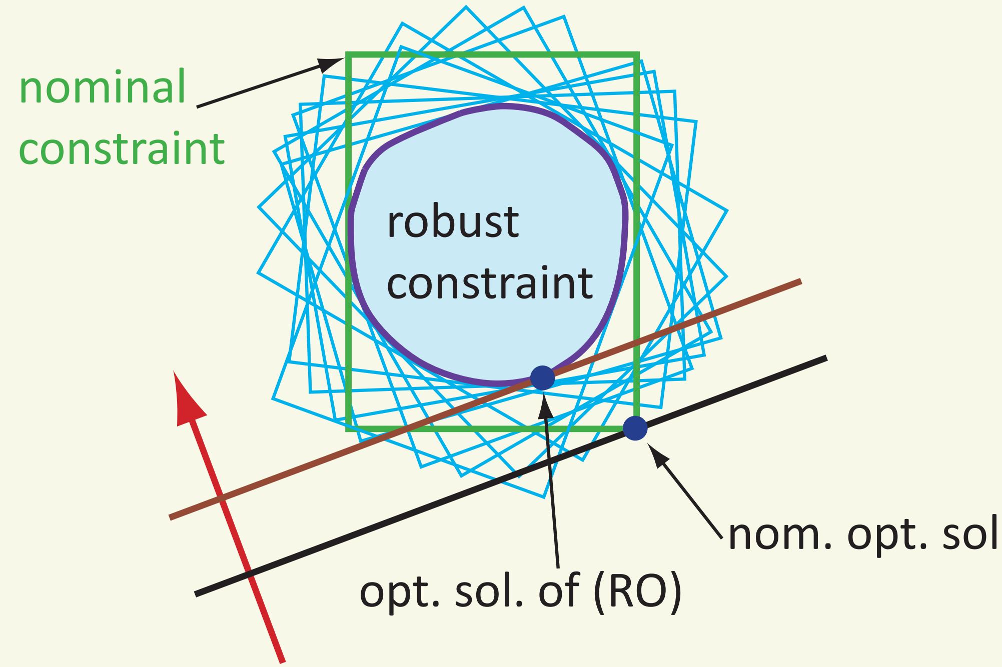


- nominal opt. prob.:

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & l_i \leq x_i \leq u_i
\end{aligned}
$$

robust opt. & conservative approx.





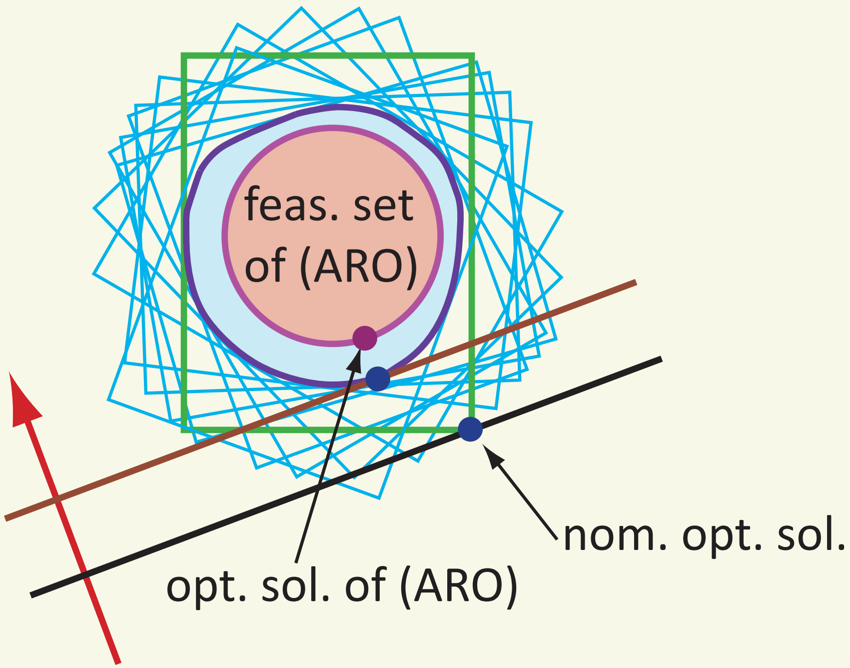
- nominal opt. prob.:

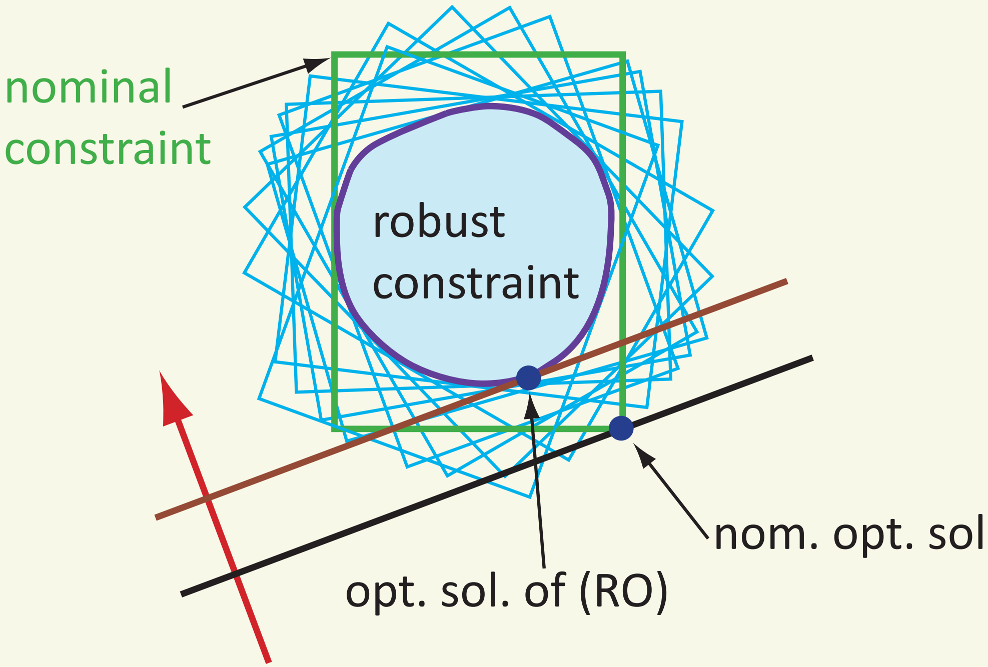
$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & l_i \le x_i \le u_i \end{aligned}$$

- robust opt. prob.:

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & l_i(\boldsymbol{z}) \le x_i \le u_i(\boldsymbol{z}) \quad (\forall \boldsymbol{z} \in \mathcal{U}) \end{aligned} \qquad \text{(RO)}$$

robust opt. & conservative approx.





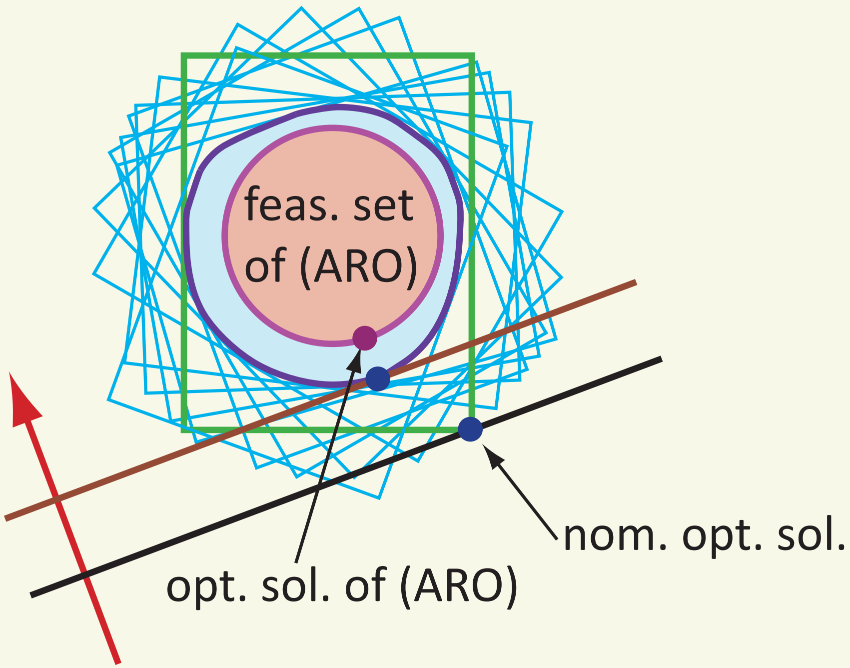
- conservative approx.:

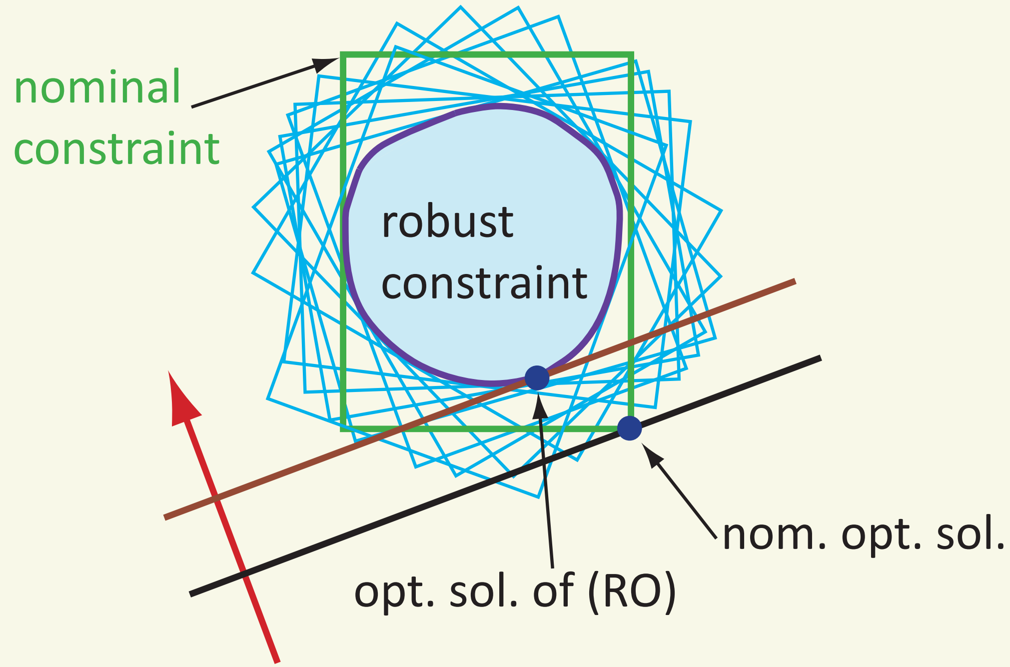
$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & S(\boldsymbol{x}) \succeq O \text{ (p.s.d.)} \end{aligned} \qquad \text{(ARO)}$$

- robust opt. prob.:

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & l_i(\boldsymbol{z}) \le x_i \le u_i(\boldsymbol{z}) \quad (\forall \boldsymbol{z} \in \mathcal{U}) \end{aligned} \qquad \text{(RO)}$$

robust opt. & conservative approx.

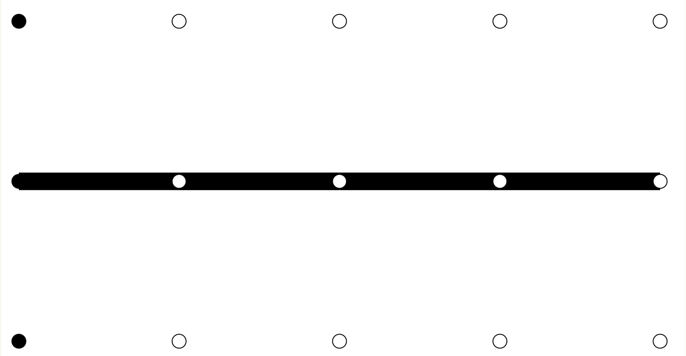




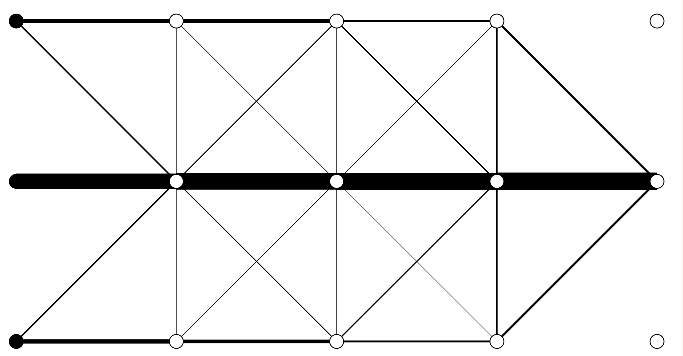
- feas. set of (ARO) $\subseteq$ feas. set of (RO)
- opt. sol. of (ARO) satisfies robust cstr.
  - but may be "less optimal" than opt. sol. of (RO)
- (ARO) is easier than (RO)

two properties of (ARO)

- stability
  - Under mild assumptions,
  - the opt. sol. of (ARO) is a stable truss.



nominal opt. sol.



opt. sol. of (ARO)

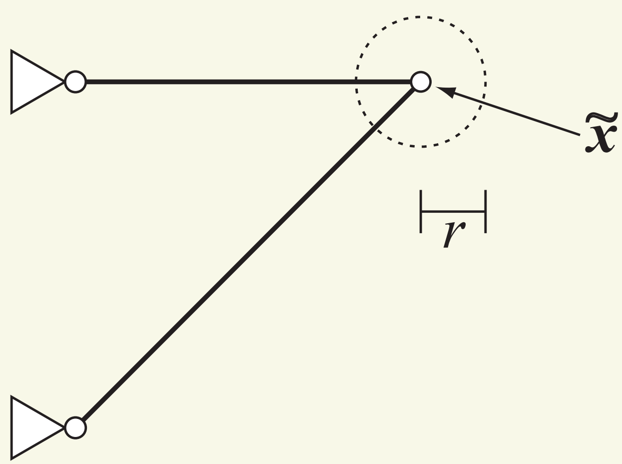
- zeroing
  - When $r = 0$, (r : level of uncertainty)
  - opt. sol. of (ARO) = nominal opt. sol.

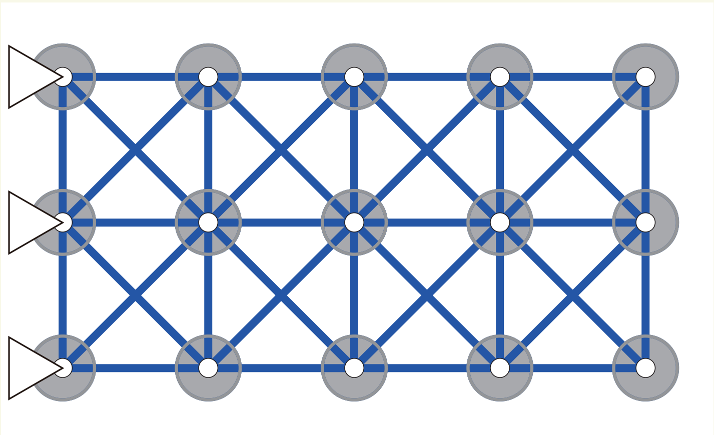
problem setting

- $\boldsymbol{x}$: locations of nodes
- uncertainty set

$$\mathcal{U}_r = \{\tilde{\boldsymbol{x}} + Az \mid \|z\| \le r\}$$

  - $\tilde{\boldsymbol{x}}$: nominal locations
  - A : constant matrix





Locations of all nodes can be uncertain.

problem setting

- $\boldsymbol{x}$: locations of nodes
- uncertainty set

$$\mathcal{U}_r = \{\tilde{\boldsymbol{x}} + Az \mid \|z\| \le r\}$$

  - $\tilde{\boldsymbol{x}}$: nominal locations
  - A : constant matrix

- nominal optimization

$$\begin{array}{ll} \min & compl(\boldsymbol{a}; \tilde{\boldsymbol{x}}) \\ \text{s. t.} & \boldsymbol{a} \ge \boldsymbol{0}, \quad vol(\boldsymbol{a}) \le \bar{V} \end{array}$$

  - $\boldsymbol{a}$: cross-sectional areas — design variables

reformulation

- nominal optimization

$$\begin{array}{ll} \min & compl(\boldsymbol{a}; \tilde{\boldsymbol{x}}) \\ \text{s. t.} & \boldsymbol{a} \geq \boldsymbol{0}, \quad vol(\boldsymbol{a}) \leq \bar{V} \end{array}$$

  - SDP formulation [Ben-Tal & Nemirovski 97]

$$\begin{array}{ll} \min & w \\ \text{s. t.} & \left[\begin{array}{c|c} w & \boldsymbol{f}^{\top} \\ \hline \boldsymbol{f} & K(\boldsymbol{a}; \tilde{\boldsymbol{x}}) \end{array}\right] \succeq O, \\ & \boldsymbol{a} \geq \boldsymbol{0}, \quad vol(\boldsymbol{a}) \leq \bar{V} \end{array}$$

$\boldsymbol{f}$: load $\quad K$: stiffness matrix

- robust optimization: robust SDP

$$\begin{array}{ll} \min & w \\ \text{s. t.} & \left[\begin{array}{c|c} w & \boldsymbol{f}^{\top} \\ \hline \boldsymbol{f} & K(\boldsymbol{a}; \boldsymbol{x}) \end{array}\right] \succeq O \quad (\forall \boldsymbol{x} \in \mathcal{U}_r) \\ & \boldsymbol{a} \geq \boldsymbol{0}, \quad vol(\boldsymbol{a}) \leq \bar{V} \end{array} \qquad \text{(RO)}$$

main result

- conservative approx. of (RO):

$$\begin{aligned}
\min \quad & w \\
\text{s. t.} \quad & \left[\begin{array}{c|c|c} & & \\ \hline & w & \boldsymbol{f}^{\top} \\ \hline & \boldsymbol{f} & \end{array}\right] + \sum_{i \in \mathcal{E}} a_i \breve{\kappa}_i \left[\begin{array}{c|c|c} & & -rB_i^{\top} \\ \hline & & \\ \hline -rB_i & & \boldsymbol{b}_i \boldsymbol{b}_i^{\top} \end{array}\right] \\
& + \left[\begin{array}{c|c|c} \mathrm{diag}(\boldsymbol{\lambda}) & & \\ \hline & & \\ \hline & & -\sum_{i \in \mathcal{E}} \lambda_i C_i C_i^{\top} \end{array}\right] \succeq O, \\
& \boldsymbol{\lambda} \geq \mathbf{0}, \quad \boldsymbol{a} \geq \mathbf{0}, \quad vol(\boldsymbol{a}) \leq \bar{V}
\end{aligned} \qquad \text{(ARO)}$$

- SDP problem — convex
  - variables: $\boldsymbol{a}$, $\boldsymbol{\lambda}$, w
  - can be solved with a primal-dual interior-point method

derivation (1/2)

- robust constraint:

$$\left[\begin{array}{c|c} w & \boldsymbol{f}^\top \\ \hline \boldsymbol{f} & K(\boldsymbol{a};\boldsymbol{x}) \end{array}\right] \succeq O \quad (\forall \boldsymbol{x} \in \mathcal{U}_r)$$

  - derive a sufficient condition

- matrix on the left-hand side:

$$\Omega_0 + \sum_{i \in \mathcal{E}} a_i \Omega_i(\boldsymbol{x})$$

  - with

$$\Omega_i(\boldsymbol{x}) = \kappa_i(\boldsymbol{x})(\hat{\boldsymbol{b}}_i + \hat{C}_i \boldsymbol{x})(\hat{\boldsymbol{b}}_i + \hat{C}_i \boldsymbol{x})^\top.$$

derivation (2/2)

$$\forall \boldsymbol{x} \in \mathcal{U}_r: \quad \Omega_0 + \sum_{i\in\mathcal{E}} a_i \Omega_i(\boldsymbol{x}) \succeq O$$

$$\Leftrightarrow \quad \forall \boldsymbol{x} \in \mathcal{U}_r, \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1}: \quad \boldsymbol{\xi}^\top \Omega_0 \boldsymbol{\xi} + \sum_{i\in\mathcal{E}} a_i \boldsymbol{\xi}^\top \Omega_i(\boldsymbol{x}) \boldsymbol{\xi} \geq 0$$

$$\Leftrightarrow \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1}: \quad \min_{\boldsymbol{x}\in\mathcal{U}_r} \Big\{ \boldsymbol{\xi}^\top \Omega_0 \boldsymbol{\xi} + \sum_{i\in\mathcal{E}} a_i \boldsymbol{\xi}^\top \Omega_i(\boldsymbol{x}) \boldsymbol{\xi} \Big\} \geq 0$$

$$\Leftarrow \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1}: \quad \boldsymbol{\xi}^\top \Omega_0 \boldsymbol{\xi} + \sum_{i\in\mathcal{E}} a_i \min_{\boldsymbol{x}\in\mathcal{U}_r} \Big\{ \boldsymbol{\xi}^\top \Omega_i(\boldsymbol{x}) \boldsymbol{\xi} \Big\} \geq 0$$

$$\Leftarrow \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1}: \quad \overbrace{\boldsymbol{\xi}^\top \Omega_0 \boldsymbol{\xi} + \sum_{i\in\mathcal{E}} a_i \breve{\kappa}_i \Big(\boldsymbol{\xi}^\top \hat{\boldsymbol{b}}_i \hat{\boldsymbol{b}}_i^\top \boldsymbol{\xi}}^{p} + 2r \min_{\eta_i\in\mathbb{R}} \{ \eta_i \hat{\boldsymbol{b}}_i^\top \boldsymbol{\xi} \mid |\eta_i| \leq \|\hat{C}_i^\top \boldsymbol{\xi}\| \} \Big) \geq 0$$

$\Leftrightarrow \quad \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1}, \forall \boldsymbol{\eta} \in \mathbb{R}^m:$ if $|\eta_i| \leq \|\hat{C}_i^\top \boldsymbol{\xi}\|$ then

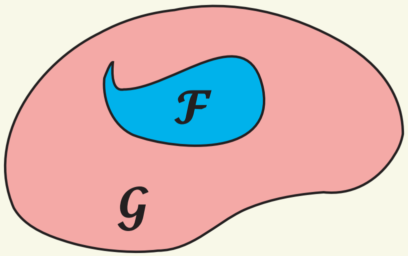
$$p - 2r a_i \breve{\kappa}_i \sum_{i\in\mathcal{E}} \eta_i \hat{\boldsymbol{b}}_i^\top \boldsymbol{\xi} \geq 0.$$

a mathematical tool

- ...for dealing with robust constraints
- $\mathcal{S}$-lemma [Pólik & Terlaky 07]: survey

a mathematical tool

- ...for dealing with robust constraints
- $\mathcal{S}$-lemma [Pólik & Terlaky 07]: survey
  - $f_i(\boldsymbol{x}),\ g(\boldsymbol{x})$: quadratic funcs.



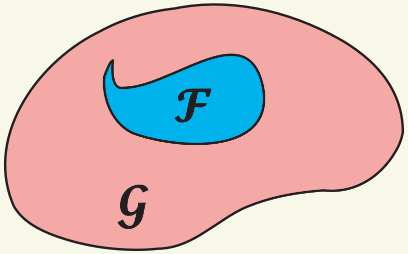
(a) $f_1(\boldsymbol{x}) \geq 0 \quad \Rightarrow \quad g(\boldsymbol{x}) \geq 0$

$\Updownarrow$

(b) $\exists \lambda_1 \geq 0 : \quad g(\boldsymbol{x}) \geq \lambda_1 f_1(\boldsymbol{x}) \ (\forall \boldsymbol{x})$

a mathematical tool

- ...for dealing with robust constraints
- $\mathcal{S}$-lemma [Pólik & Terlaky 07]: survey
  - $f_i(\boldsymbol{x}),\ g(\boldsymbol{x})$: quadratic funcs.



(a) $f_1(\boldsymbol{x}) \geq 0 \quad \Rightarrow \quad g(\boldsymbol{x}) \geq 0$

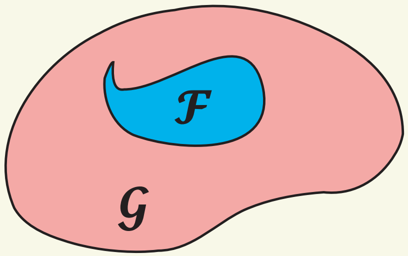
$\Updownarrow$

(b) $\exists \lambda_1 \geq 0 : \quad g(\boldsymbol{x}) \geq \lambda_1 f_1(\boldsymbol{x}) \ (\forall \boldsymbol{x})$

- Farkas' lemma
  - $f_i(\boldsymbol{x}),\ g(\boldsymbol{x})$: linear funcs.
  - (a') : "$f_1(\boldsymbol{x}) \geq 0,\ g(\boldsymbol{x}) < 0$" has no solution.

a mathematical tool

- ...for dealing with robust constraints
- $\mathcal{S}$-lemma   [Pólik & Terlaky 07]: survey
  - $f_i(\boldsymbol{x})$, $g(\boldsymbol{x})$: quadratic funcs.



(a) $\underbrace{f_1(\boldsymbol{x}) \geq 0, \dots, f_m(\boldsymbol{x}) \geq 0}_{\mathcal{F}} \quad \Rightarrow \quad \underbrace{g(\boldsymbol{x}) \geq 0}_{\mathcal{G}}$

$\Uparrow$

(b) $\exists \lambda_i \geq 0 : \quad g(\boldsymbol{x}) \geq \lambda_1 f_1(\boldsymbol{x}) + \cdots + \lambda_m f_m(\boldsymbol{x}) \ (\forall \boldsymbol{x})$

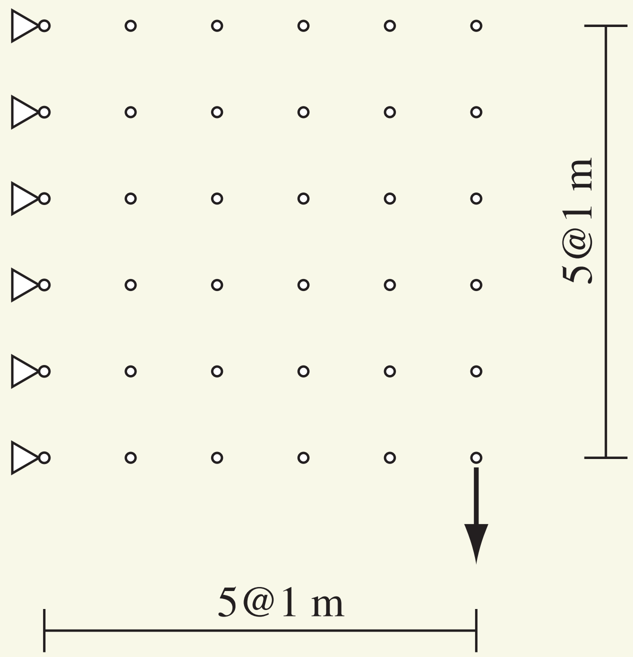
- (a) : $\mathcal{F} \subseteq \mathcal{G}$

  ($\mathcal{F}$: variation of compliance; $\mathcal{G}$: feasible set)

- (b) : p.s.d. constraint on a matrix

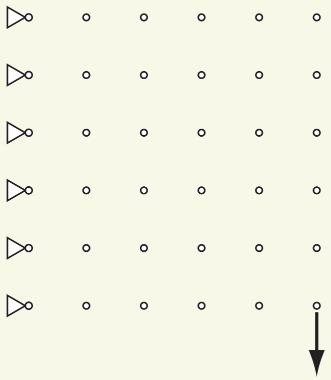
  $\rightarrow$ constraint of SDP

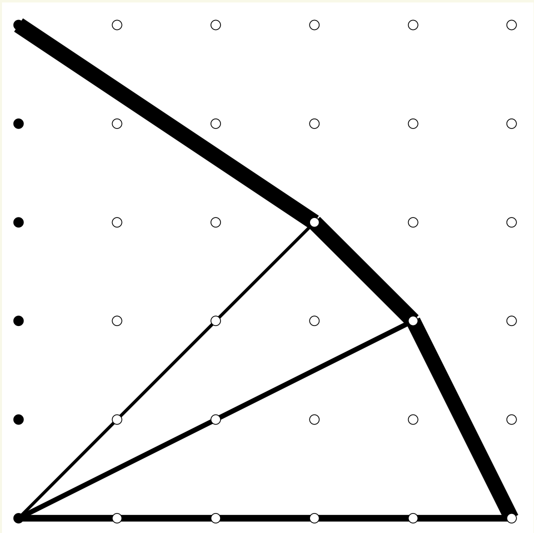
ex.) 418-bar truss



- ground structure
  - Any two nodes are connected by a member.
  - Overlapping members are removed.
- uncertainty: locations of all nodes
  - incl. supports

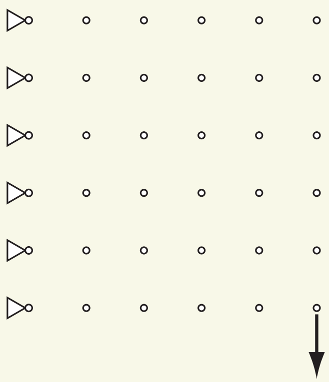
ex.) 418-bar truss

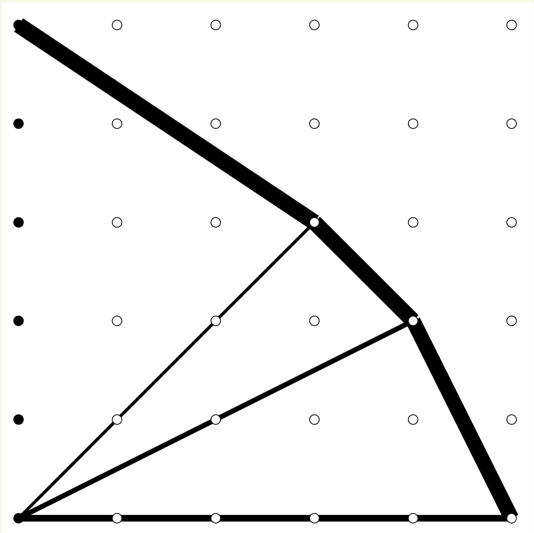




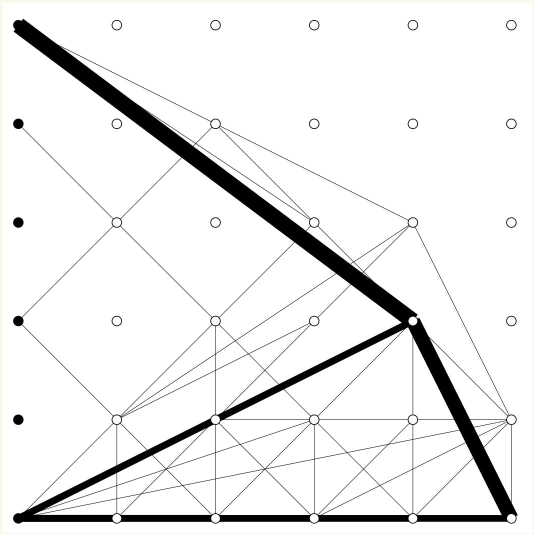
nominal opt. sol.

ex.) 418-bar truss





nominal opt. sol.

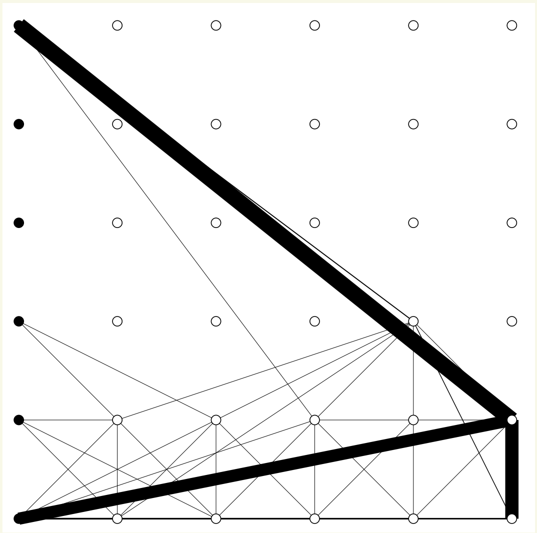


$r = 0.02\,\mathrm{m}$

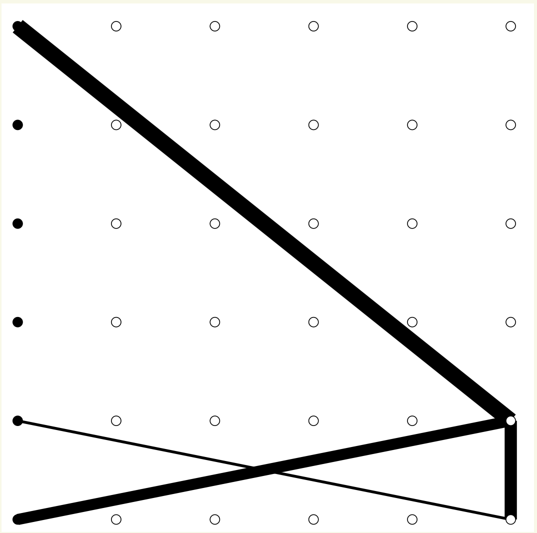
very large r

↓

few nodes

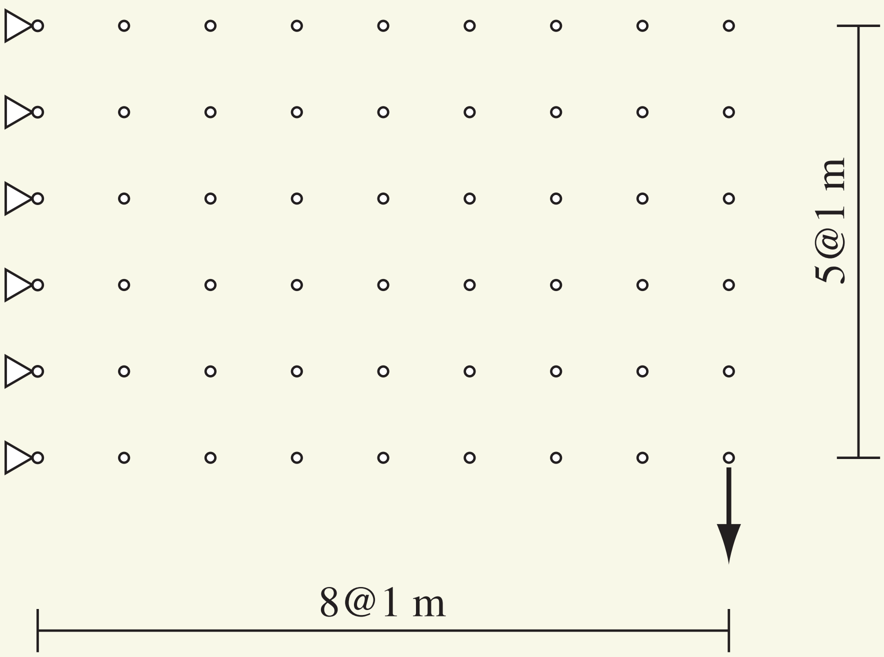


$r = 0.05\,\mathrm{m}$



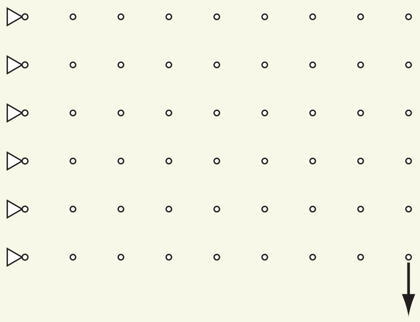
$r = 0.1\,\mathrm{m}$

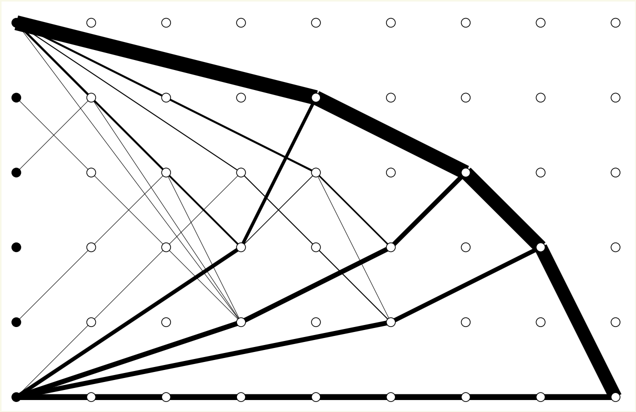
ex.) 919-bar truss



- uncertainty: locations of all nodes
  - incl. supports

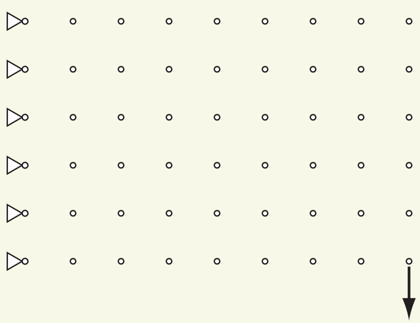
ex.) 919-bar truss

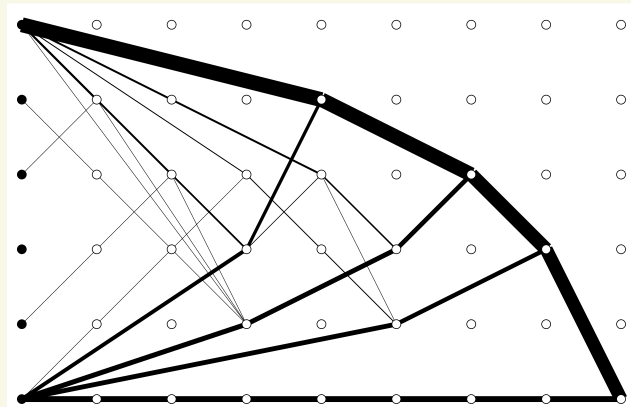




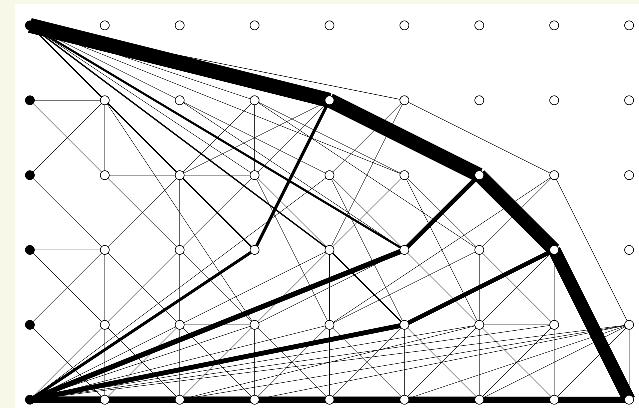
nominal opt.

ex.) 919-bar truss





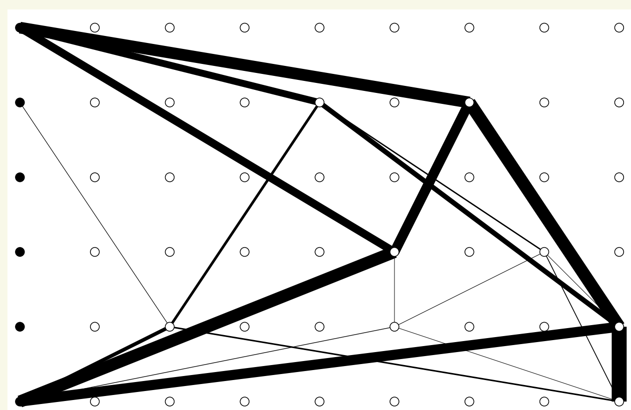
nominal opt.



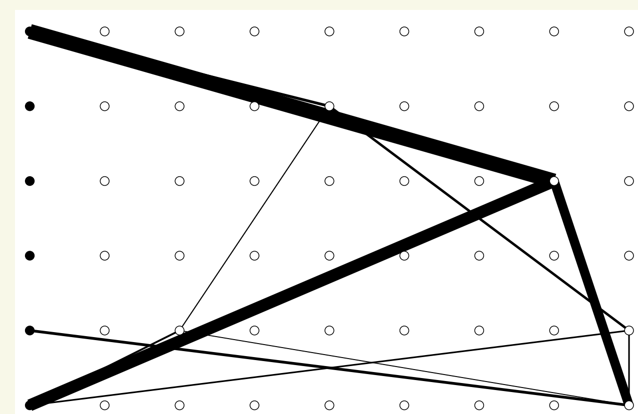
$r = 0.02\,\mathrm{m}$

r (m)	Time (s)
0	10.2
0.02	1285.2
0.05	863.2
0.10	685.8

(SeDuMi 1.3)



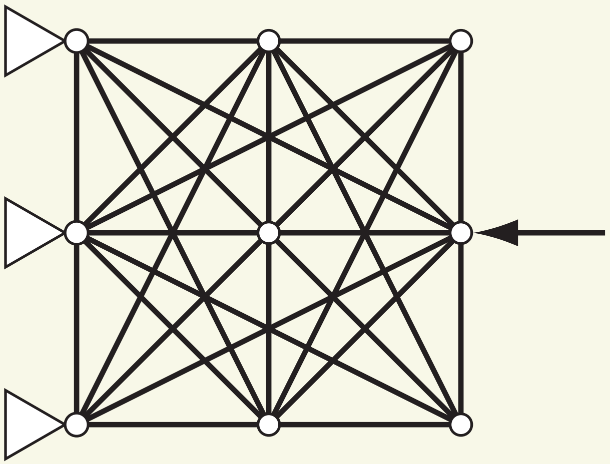
$r = 0.05\,\mathrm{m}$



$r = 0.1\,\mathrm{m}$

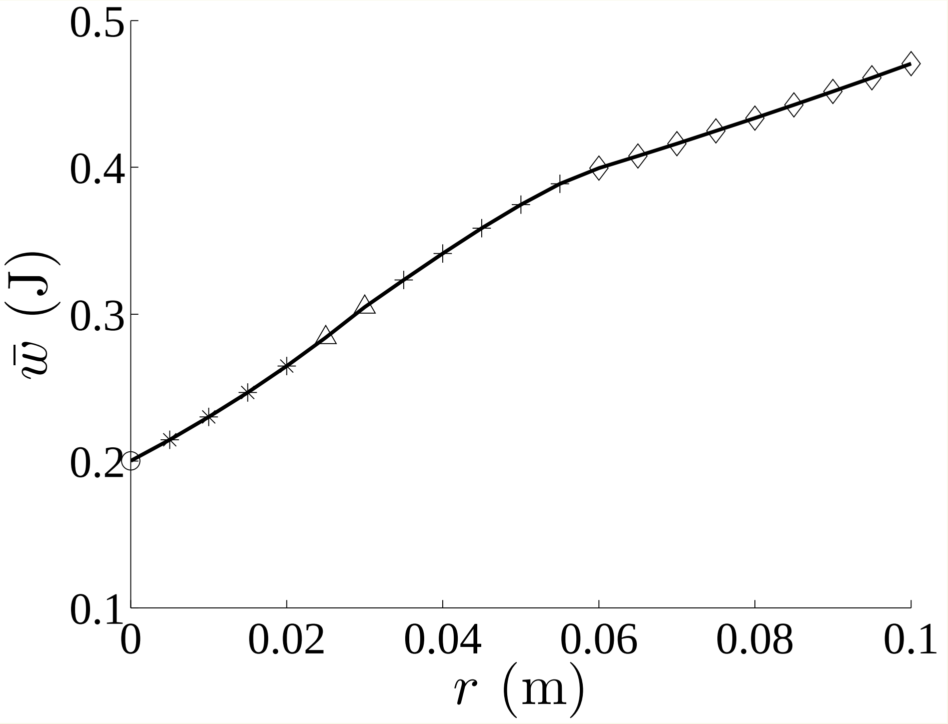
ex.) effect of uncertainty magnitude

- level of uncertainty: r
  - 0 (= no uncertainty) $\leftrightarrow$ $0.1\,\mathrm{m}$ ($\simeq$ 10% uncertainty)
- opt. val. of SDP: $\bar{w}$ ($\simeq$ worst compliance)



ex.) effect of uncertainty magnitude

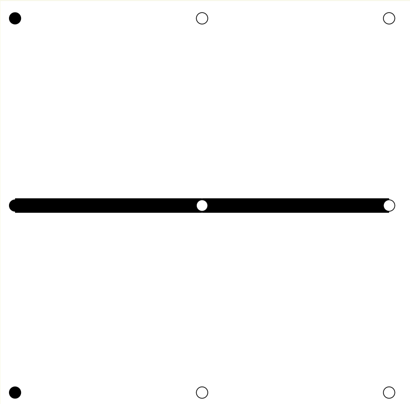
- level of uncertainty: r
  - 0 (= no uncertainty) $\leftrightarrow$ $0.1\,\mathrm{m}$ ($\simeq$ 10% uncertainty)
- opt. val. of SDP: $\bar{w}$ ($\simeq$ worst compliance)



- robustness curve (trade off)
  - performance requirement ↑ (i.e., $\bar{w}$ ↓) $\leftrightarrow$ robustness r ↓

ex.) effect of uncertainty magnitude

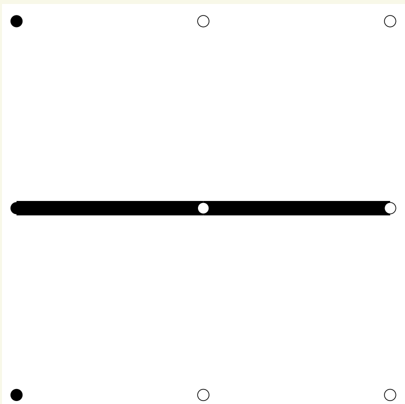
- level of uncertainty: r vs. variation of topology



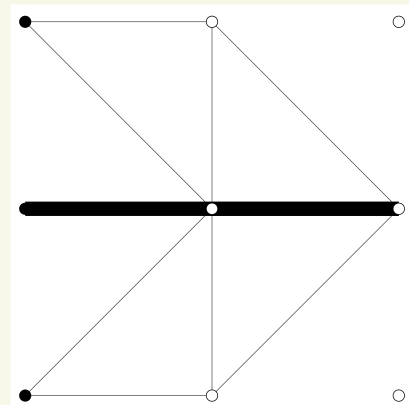
$r = 0$

ex.) effect of uncertainty magnitude

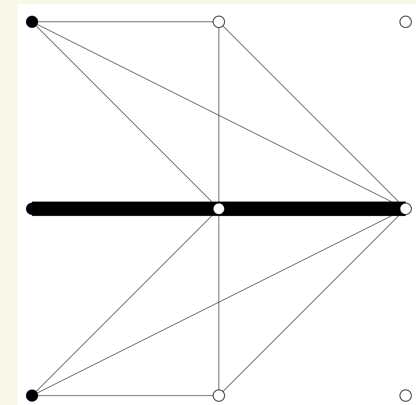
- level of uncertainty: r vs. variation of topology



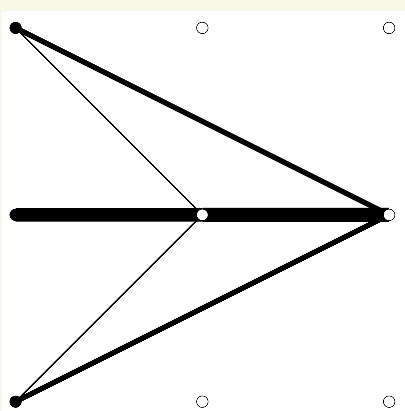
$r = 0$



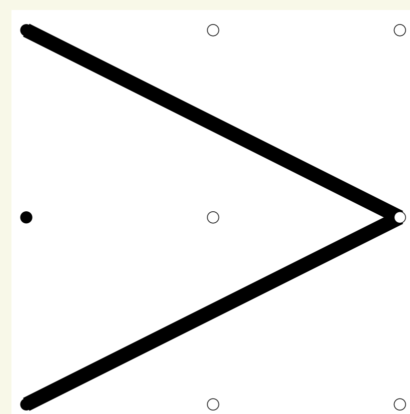
$0.05 \leq r \leq 0.02$



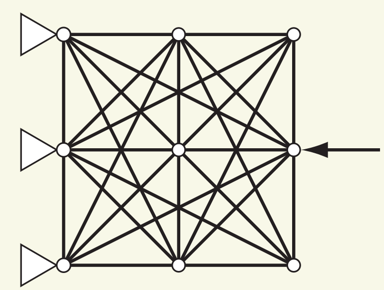
$0.025 \leq r \leq 0.03$



$0.035 \leq r \leq 0.055$



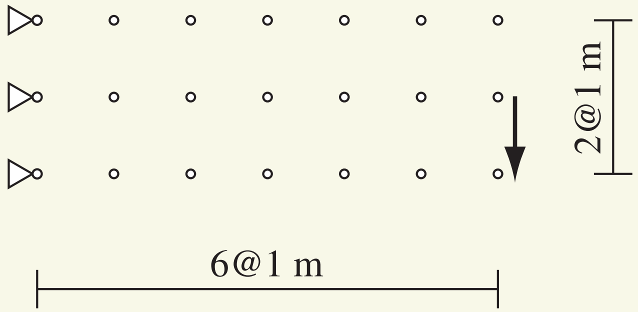
$0.06 \leq r \leq 0.1$

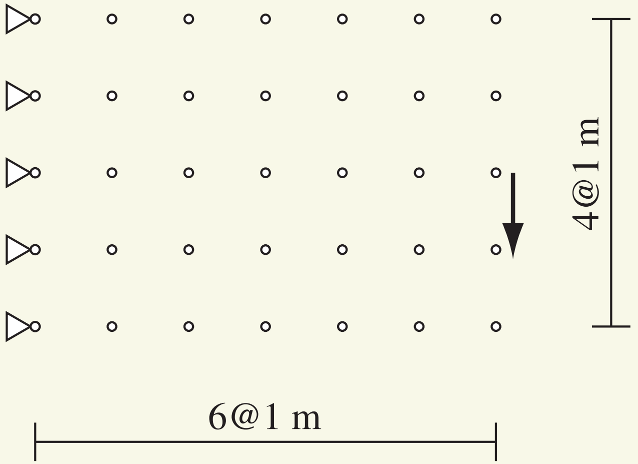


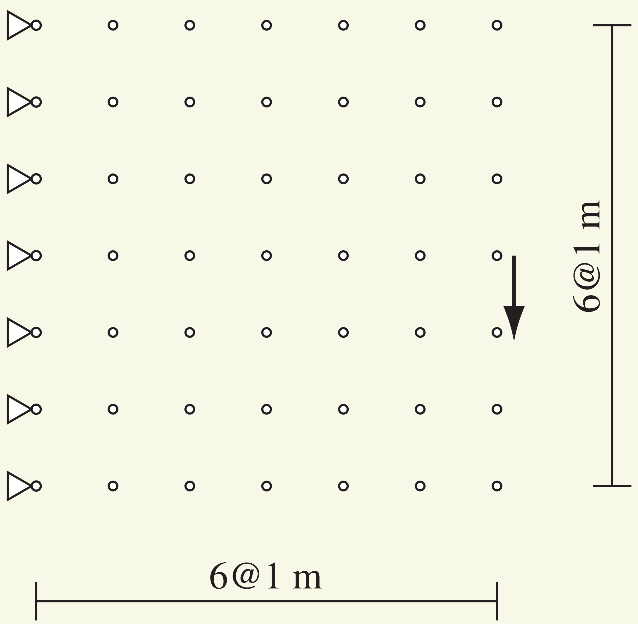
- large r $\leftrightarrow$ fewer nodes

ex.) more examples

- ground structures

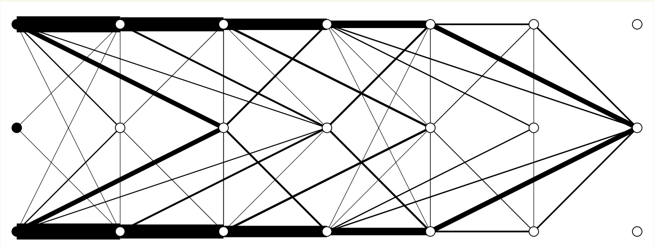


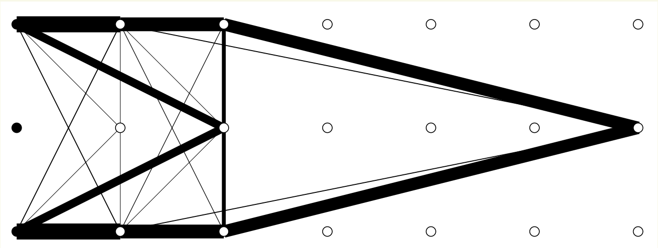


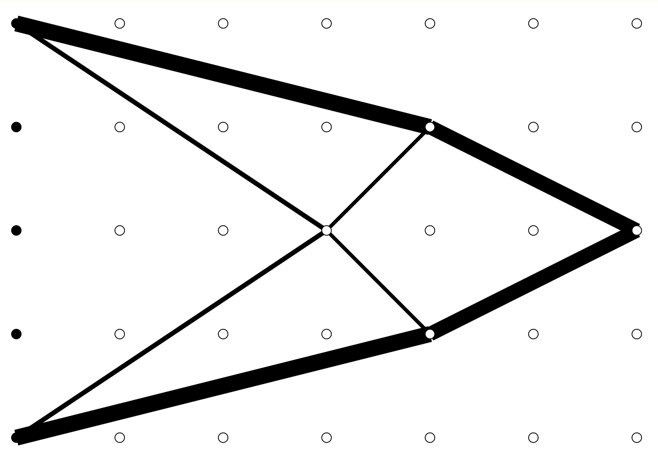


ex.) more examples

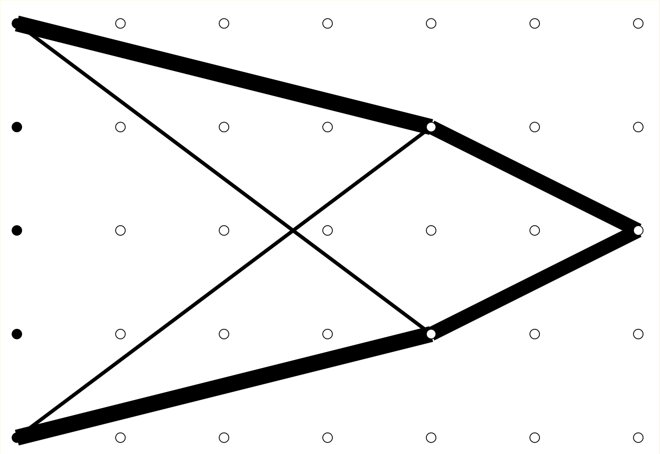
- solutions







nominal opt.

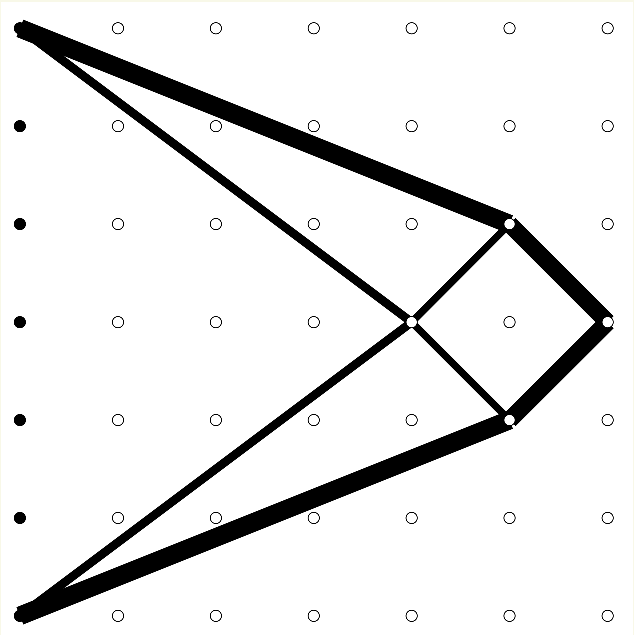


robust opt. ($r = 0.05\,\mathrm{m}$)

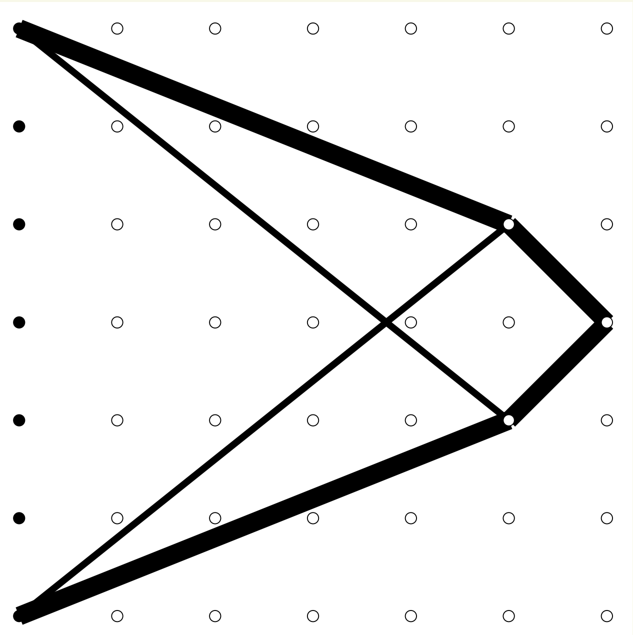
- Nominal opt. sol. is also stable.
- Robust opt. sol. has less nodes.

ex.) more examples

- solutions



nominal opt.



robust opt. ($r = 0.05\,\mathrm{m}$)

- Nominal opt. sol. is also stable.
- Robust opt. sol. has less nodes.

conclusions

- robust truss optimization
  - uncertainty: locations of nodes $\boldsymbol{x} = \tilde{\boldsymbol{x}} + A\boldsymbol{z}$ ($\|\boldsymbol{z}\| \leq r$)
  - worst-case compliance $\rightarrow$ Min.
- SDP (semidefinite programming) approach:
  - conservative approximation
    - (SDP) $\subseteq$ (RO)
    - providing an upper bound
  - convex optimization
    - primal-dual interior-point method
  - stability of the optimal solution
  - zeroing