A Fast First-Order Optimization Approach to Quasistatic Elastoplastic Analysis with von Mises Yield Criterion

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• equilibrium analysis w/ von Mises criterion ≈ group LASSO
  • optimization approach to computational plasticity
    • min. potential energy
    [Maier ’68, ’70], etc.
• data science
  • needs for solving large-scale (convex) optimization
  • least squares w/ regularization
    • LASSO [Tibshirani ’96]    group LASSO [Yuan & Lin ’06]
• fast 1st-order algo. for (convex) optim.
  • accelerated gradient method
    [Nesterov ’83]
  • accelerated proximal grad. meth.
    [Beck & Teboulle ’09]
equilibrium analysis w/ von Mises criterion \( \approx \) group LASSO

- optimization approach to computational plasticity
  - min. potential energy (!)

- data science
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    - LASSO [Tibshirani ’96] group LASSO [Yuan & Lin ’06] (!)

- fast 1st-order algo. for (convex) optim.
  - accelerated gradient method [Nesterov ’83]
  - accelerated proximal grad. meth. (!) [Beck & Teboulle ’09]

(accelerated) proximal gradient method

- 1st-order meth. for convex optim. [Bruck ’77], [Passty ’79], etc.
  - generalization of projected gradient method
- can solve nonsmooth problems
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application to $\ell_1$-regularized least squares (LASSO)
- ISTA (iterative shrinkage-thresholding algorithms)
  [Chambolle, DeVore, Lee, & Lucier ’98], [Daubechies, Defrise, & Mol ’04]
  - $O(1/k)$ convergence in obj. value ($k$: iteration counter)
- acceleration: FISTA (fast ISTA) [Beck & Teboulle ’09]
  - $O(1/k^2)$

- restart of acceleration [O’Donoghue & Candes ’15]
  - monotonic decrease of obj. value
proximal gradient method

- convex optimization:

  \[
  \text{Minimize } f(x) + g(x)
  \]

- \( f \) : convex, differentiable  \( g \) : convex
proximal gradient method

- convex optimization:
  
  \[
  \text{Minimize } f(x) + g(x)
  \]

- \( f \): convex, differentiable \( g \): convex

- iteration:
  
  \[
  x^{(k+1)} := \text{prox}_{\alpha g}(x^{(k)} - \alpha \nabla f(x^{(k)}))
  \]

- def. of proximal mapping:
  
  \[
  \text{prox}_{\alpha g}(x) = \arg \min_{z} \left\{ \alpha g(z) + \frac{1}{2} \| z - x \|^{2} \right\}
  \]

- \( \alpha \in (0, 1/L] \): step size \( (L \): Lipschitz constant of \( \nabla f \) )

- useful if computation of \( \text{prox}_{\alpha g} \) is easy
  
  \( \simeq \) if \( g \) has a simple form
accelerated proximal gradient method

• original version — $O(1/k)$:

\[ x^{(k+1)} := \text{prox}_{\alpha g} (x^{(k)} - \alpha \nabla f(x^{(k)})) \]

• accelerate version — $O(1/k^2)$: [Beck & Teboulle '09]

\[ x^{(k+1)} := \text{prox}_{\alpha g} (y^{(k)} - \alpha \nabla f(y^{(k)})) \]
\[ y^{(k+1)} := x^{(k+1)} + \omega^{(k)} (x^{(k+1)} - x^{(k)}) \]

• e.g., $\omega^{(k)} := k/(k + 3)$
accelerated proximal gradient method

- original version — $O(1/k)$:
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  \]

- accelerate version — $O(1/k^2)$: [Beck & Teboulle ’09]
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  x^{(k+1)} := \text{prox}_{\alpha g} (y^{(k)} - \alpha \nabla f(y^{(k)}))
  
  y^{(k+1)} := x^{(k+1)} + \omega^{(k)} (x^{(k+1)} - x^{(k)})
  \]

- restart (reset $\omega^{(k)} := 0$) — monotonicity [O’Donoghue & Candes ’15]
accelerated proximal gradient method

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$$x^{(k+1)} := \text{prox}_{\alpha g}(x^{(k)} - \alpha \nabla f(x^{(k)}))$$

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$$x^{(k+1)} := \text{prox}_{\alpha g}(y^{(k)} - \alpha \nabla f(y^{(k)}))$$

$$y^{(k+1)} := x^{(k+1)} + \omega^{(k)}(x^{(k+1)} - x^{(k)})$$

• restart (reset $\omega^{(k)} := 0$) — monotonicity [O’Donoghue & Candes ’15]

• easy to implement

• fast convergence

• applicable to large-scale problems

• most of computation: matrix-vector products

no system of linear eqs.
accelerated proximal gradient method

- original version — $O(1/k)$:

  \[
  x^{(k+1)} := \text{prox}_{\alpha g}(x^{(k)} - \alpha \nabla f(x^{(k)}))
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- accelerate version — $O(1/k^2)$: [Beck & Teboulle ’09]

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  x^{(k+1)} := \text{prox}_{\alpha g}(y^{(k)} - \alpha \nabla f(y^{(k)})) \\
  y^{(k+1)} := x^{(k+1)} + \omega^{(k)}(x^{(k+1)} - x^{(k)})
  \]

- restart (reset $\omega^{(k)} := 0$) — monotonicity [O’Donoghue & Candès ’15]

- “Unfortunately, it is very difficult to obtain strong intuition about the mechanism by which this remarkable phenomenon occurs.”

LASSO (least absolute shrinkage and selection operator)

- solves regularized least squares: $\text{Minimize } \frac{1}{2} \|Ax - b\|_2^2 + \gamma \sum_{j=1}^{n} |x_j|$

  $\gamma > 0$ : parameter

- to find a sparse solution to a linear regression problem
LASSO (least absolute shrinkage and selection operator)

- solves regularized least squares:  
  \[
  \text{Minimize } \frac{1}{2} \|Ax - b\|_2^2 + \gamma \sum_{j=1}^{n} |x_j| \\
  f(x) + g(x) \quad \gamma > 0 : \text{parameter}
  \]

- to find a sparse solution to a linear regression problem

- ISTA = a proximal gradient method for LASSO:
  \[
  x^{(k+1)} = \text{prox}_{\alpha g}(x^{(k)} - \alpha \nabla f(x^{(k)}))
  \]

- \( \text{prox}_{\alpha g} \) is easily computed as
  \[
  \text{prox}_{\alpha g}(s) = (s - \alpha 1)_+ - (-s - \alpha 1)_+
  \]

- FISTA (fast ISTA) = accelerated ISTA  
  [Beck & Teboulle '09]
**group LASSO**

- **LASSO:**
  
  $\text{Minimize } \frac{1}{2} \| Ax - b \|_2^2 + \gamma \sum_{j=1}^{n} |x_j|$

  - attempts to zero many entries $x_j$
  - $\sum_{j=1}^{n} |x_j| : \ell_1$-norm of $(x_1, \ldots, x_n)$

- **group LASSO ($\simeq$ von Mises):**

  $\text{Minimize } \frac{1}{2} \| Ax - b \|_2^2 + \gamma \sum_{l=1}^{m} \|x_l\|_2$

  - attempts to zero many sub-vectors $x_l$
  - $\sum_{l=1}^{m} \|x_l\|_2 : \ell_1$-norm of $(\|x_1\|_2, \ldots, \|x_m\|_2)$
  - proximal grad. meth. for group LASSO

[Tibshirani '96]

[Yuan & Lin '06]

[Mosci, Rosasco, & Santoro '10]
- incremental problem
- optimization-based approach — potential energy minimization
  - SOCP (second-order cone programming):
    - Minimize (convex quad. fcn.)
    - subject to (second-order cones) & (linear eqs.)
  - can be solved with a primal-dual interior-point method

\[ s_0 \geq \| (s_{11}, s_{12}) \|_2 \]
equilibrium analysis w/ von Mises criterion

- incremental problem
- optimization-based approach — potential energy minimization
  - SOCP (second-order cone programming):
    
    \[
    \text{Minimize } \quad \text{(convex quad. fcn.)} \\
    \text{subject to } \quad \text{(second-order cones) } \& \text{ (linear eqs.)}
    \]
    
    [Bisbos, Makrodimopoulos, & Pardalos ’05], [Yonekura & K. ’12]
  - can be solved with a primal-dual interior-point method

- equivalent formulation:
  
  \[
  \text{Minimize } \quad \text{(nonsmooth convex fcn.)}
  \]

- unconstrained optimization
- suitable for an accelerated proximal gradient method
  - (potentially) efficient for large-scale problems
basic assumptions

- small deformation
  - strain decomposition $\varepsilon = \varepsilon^e + \varepsilon^p$

- strain hardening
  - linear isotropic
  - linear kinematic

- incremental problem
convex, nonsmooth, unconstrained optimization:

\[
\begin{align*}
\text{Min. } & \sum_{i=1}^{m} \frac{1}{2} \varepsilon_i^e : C_i : \varepsilon_i^e + \sum_{i=1}^{m} \left( \sqrt{\frac{2}{3}} R_i \| \varepsilon_i^p \|_F + \frac{1}{3} \| \varepsilon_i^p \|_F^2 \right) - f^T u
\end{align*}
\]

- elastic energy
- plastic dissipation

- variables: \( u \) (inc. disp.), \( \varepsilon_i^p \) (inc. plastic strain)

- \( \varepsilon_i^e \) (inc. elastic strain) can be eliminated.

- substitute \( \varepsilon_i^e = B_i \cdot u - \varepsilon_i^p \)
a formulation of incremental problem

- convex, nonsmooth, unconstrained optimization:

\[
\text{Min. } \sum_{i=1}^{m} \frac{1}{2} \varepsilon_i^e : C_i : \varepsilon_i^e + \sum_{i=1}^{m} \left( \sqrt{\frac{2}{3}} R_i \| \varepsilon_i^p \|_F + \frac{1}{3} \| \varepsilon_i^p \|_F^2 \right) - f^\top u \quad (\blacklozenge)
\]

\( \varepsilon_i^e \) elastic energy

\( \varepsilon_i^p \) plastic dissipation

- variables: \( u \) (inc. disp.), \( \varepsilon_i^p \) (inc. plastic strain) \( \varepsilon_i^e := \varepsilon_i^e(u, \varepsilon_i^p) \)

- apply proximal gradient method to:

\[
(\blacklozenge) \iff \text{Min. } f(u, \varepsilon^p) + g(\varepsilon^p) \quad \text{w/ } g(\varepsilon^p) = \sum_{i=1}^{m} \sqrt{\frac{2}{3}} R_i \| \varepsilon_i^p \|_F
\]

- \( f \) : convex quadratic

- \( g \) : convex, nonsmooth, & simple

- \( \approx \) group LASSO:

\[
\text{Min. } \frac{1}{2} \| Ax - b \|_2^2 + \gamma \sum_{l=1}^{m} \| x_l \|_2
\]
• iteration of prox. grad. meth.:

\[
\begin{bmatrix}
  u^{(k+1)} \\
  p
\end{bmatrix} :=
\begin{bmatrix}
  u^{(k)} \\
  \varepsilon^{p(k)}
\end{bmatrix} - \alpha \nabla^2 f(u, \varepsilon) \begin{bmatrix}
  u^{(k)} \\
  \varepsilon^{p(k)}
\end{bmatrix}
\]

\[
\varepsilon^{p(k+1)}_i := \begin{cases}
  0 & \text{if } \|p_i\| \leq \alpha \sqrt{\frac{2}{3}R_i} \\
  \left(\|p_i\| - \alpha \sqrt{\frac{2}{3}R_i}\right) \frac{p_i}{\|p_i\|} & \text{otherwise}
\end{cases}
\quad (\forall i)
\]

• \(\nabla^2 f(u, \varepsilon)\) : a sparse constant matrix

• \(p\) : intermediate variable (for notational simplicity)

• very cheap computation per iteration

• only small modification to achieve acceleration
preliminary numerical experiments

- perforated plate in 3D

- comparison
  - primal-dual interior-point method \cite{Tutuncu, Toh, Todd '09} 
    for second-order cone programming (SOCP): SDPT3 (ver. 4)
    - solves standard form of SOCP.
  - proposed method
    - solves unconstrained nonsmooth convex opt.
    - Matlab implementation
ex.) convergence history

- proximal gradient methods

- "---" unaccelerated
- "......" accelerated, w/o restart
- "——" accelerated, w/ restart
ex.) computational time

- time vs. DOF
- “×” SDPT3 (interior-point method)
- “○” proposed method
ex.) computational accuracy

- objective value
  - $(\text{APGM: proposed}) < (\text{SDPT3})$

- relative difference $= \frac{(\text{SDPT3}) - (\text{APGM})}{(\text{SDPT3})}$
conclusions

- incremental elastoplastic analysis w/ von Mises criterion
  - 2nd-order cone prog. (SOCP) + intr.-point meth. (IPM) (exist)
  - unconstrained nonsmooth convex optimization (new)
    - (convex quadratic fcn.) + (sum of $\ell_2$-norms)
    - $\approx$ group LASSO (a regularized least squares)

- accelerated proximal gradient method
  - fast convergence: $O(1/k^2)$
  - cheap computational cost (no system of linear eqs.)
    - fut. wrk.: interpret update & acceleration schemes
  - faster than a standard IPM
    - fut. wrk.: comparison with, e.g., a return-mapping meth.