A Fast First-Order Optimization Approach to Quasistatic Elastoplastic Analysis with von Mises Yield Criterion

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theme

- equilibrium analysis w/ von Mises criterion $\simeq$ group LASSO
  - optimization approach to computational plasticity
    - min. potential energy [Maier '68, '70], etc.
  - data science
    - needs for solving large-scale (convex) optimization
    - least squares w/ regularization
      - LASSO [Tibshirani '96] group LASSO [Yuan & Lin '06]
  - fast 1st-order algo. for (convex) optim.
    - accelerated gradient method [Nesterov '83]
    - accelerated proximal grad. meth. [Beck & Teboulle '09]
  - Y. K.: "A fast first-order optimization approach to elastoplastic analysis of skeletal structures." *Optim. Eng.*, **17**, 861–896 (2016).

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(accelerated) proximal gradient method

- 1st-order meth. for convex optim. [Bruck '77], [Passty '79], etc.
  - generalization of projected gradient method
  - can solve nonsmooth problems

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- 1st-order meth. for convex optim. [Bruck '77], [Passty '79], etc.
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  - can solve nonsmooth problems
- application to ℓ_1-regularized least squares (LASSO)
  - ISTA (iterative shrinkage-thresholding algorithms)
    [Chambolle, DeVore, Lee, & Lucier '98], [Daubechies, Defrise, & Mol '04]
    - $O(1/k)$ convergence in obj. value (k: iteration counter)
  - acceleration: FISTA (fast ISTA) [Beck & Teboulle '09]
    - $O(1/k^2)$
  - restart of acceleration [O'Donoghue & Candes '15]
    - monotonic decrease of obj. value

proximal gradient method

- convex optimization:

$$\text{Minimize} \quad f(\boldsymbol{x}) + g(\boldsymbol{x})$$

  - f : convex, differentiable g : convex

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- iteration:

$$\boldsymbol{x}^{(k+1)} := \mathbf{prox}_{\alpha g}(\boldsymbol{x}^{(k)} - \alpha \nabla f(\boldsymbol{x}^{(k)}))$$

  - def. of proximal mapping:

$$\mathbf{prox}_{\alpha g}(\boldsymbol{x}) = \arg\min_{\boldsymbol{z}} \left\{ \alpha g(\boldsymbol{z}) + \frac{1}{2}\|\boldsymbol{z} - \boldsymbol{x}\|^2 \right\}$$

  - $\alpha \in (0, 1/L]$: step size (L : Lipschitz constant of ∇f)
  - useful if computation of $\mathbf{prox}_{\alpha g}$ is easy
    $\simeq$ if g has a simple form

accelerated proximal gradient method

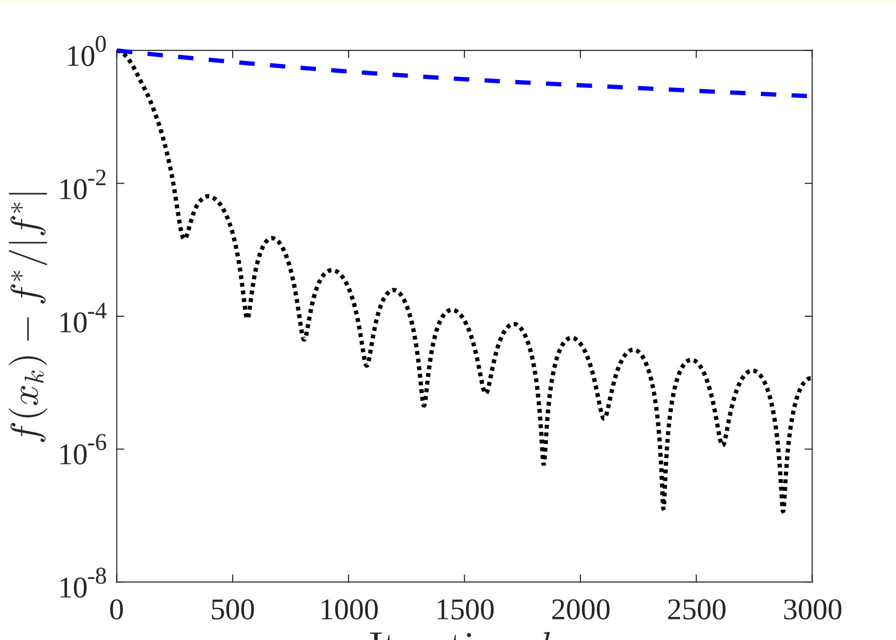
- original version — $O(1/k)$:

$$\boldsymbol{x}^{(k+1)} := \mathbf{prox}_{\alpha g}(\boldsymbol{x}^{(k)} - \alpha \nabla f(\boldsymbol{x}^{(k)}))$$

- accelerate version — $O(1/k^2)$: [Beck & Teboulle '09]

$$\boldsymbol{x}^{(k+1)} := \mathbf{prox}_{\alpha g}(\boldsymbol{y}^{(k)} - \alpha \nabla f(\boldsymbol{y}^{(k)}))$$
$$\boldsymbol{y}^{(k+1)} := \boldsymbol{x}^{(k+1)} + \omega^{(k)}(\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)})$$

  - e.g., $\omega^{(k)} := k/(k+3)$



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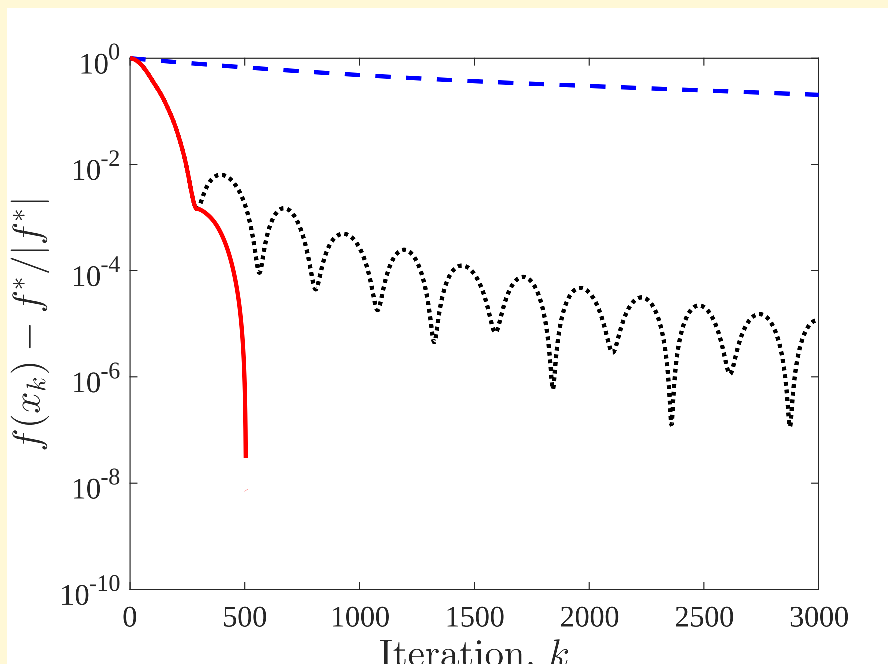
- restart (reset $\omega^{(k)} := 0$) — monotonicity [O'Donoghue & Candes '15]



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- restart (reset $\omega^{(k)} := 0$) — monotonicity [O'Donoghue & Candes '15]
  - easy to implement
  - fast convergence
  - applicable to large-scale problems
    - most of computation: matrix-vector products
      no system of linear eqs.

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- restart (reset $\omega^{(k)} := 0$) — monotonicity [O'Donoghue & Candes '15]
- "Unfortunately, it is very difficult to obtain strong intuition about the mechanism by which this remarkable phenomenon occurs."
  D. P. Bertsekas: *Convex Optimization Algorithms*, p. 322.

LASSO (least absolute shrinkage and selection operator)

- solves regularized least squares: [Tibshirani '96]

$$\text{Minimize} \quad \frac{1}{2}\|A\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \gamma \sum_{j=1}^{n} |x_j|$$

$\gamma > 0$: parameter

- to find a sparse solution to a linear regression problem

LASSO (least absolute shrinkage and selection operator)

- solves regularized least squares: [Tibshirani '96]

$$\text{Minimize} \quad \underbrace{\frac{1}{2}\|A\boldsymbol{x} - \boldsymbol{b}\|_2^2}_{f(\boldsymbol{x})} + \underbrace{\gamma \sum_{j=1}^{n} |x_j|}_{g(\boldsymbol{x})}$$

$\gamma > 0$: parameter

  - to find a sparse solution to a linear regression problem

- ISTA = a proximal gradient method for LASSO:

$$\boldsymbol{x}^{(k+1)} = \mathbf{prox}_{\alpha g}(\boldsymbol{x}^{(k)} - \alpha \nabla f(\boldsymbol{x}^{(k)}))$$

  - $\mathbf{prox}_{\alpha g}$ is easily computed as

$$\mathbf{prox}_{\alpha g}(\boldsymbol{s}) = (\boldsymbol{s} - \alpha \mathbf{1})_+ - (-\boldsymbol{s} - \alpha \mathbf{1})_+$$

- FISTA (fast ISTA) = accelerated ISTA [Beck & Teboulle '09]

group LASSO

- LASSO: [Tibshirani '96]

$$\text{Minimize} \quad \frac{1}{2}\|A\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \gamma \sum_{j=1}^{n} |x_j|$$

  - attempts to zero many entries x_j
  - $\sum_{j=1}^{n} |x_j|$: ℓ_1-norm of $(x_1, \dots, x_n)$

- group LASSO (≃ von Mises): [Yuan & Lin '06]

$$\text{Minimize} \quad \frac{1}{2}\|A\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \gamma \sum_{l=1}^{m} \|\boldsymbol{x}_l\|_2$$

  - attempts to zero many sub-vectors $\boldsymbol{x}_l$
  - $\sum_{l=1}^{m} \|\boldsymbol{x}_l\|_2$: ℓ_1-norm of $(\|\boldsymbol{x}_1\|_2, \dots, \|\boldsymbol{x}_m\|_2)$
  - proximal grad. meth. for group LASSO

[Mosci, Rosasco, & Santoro '10]

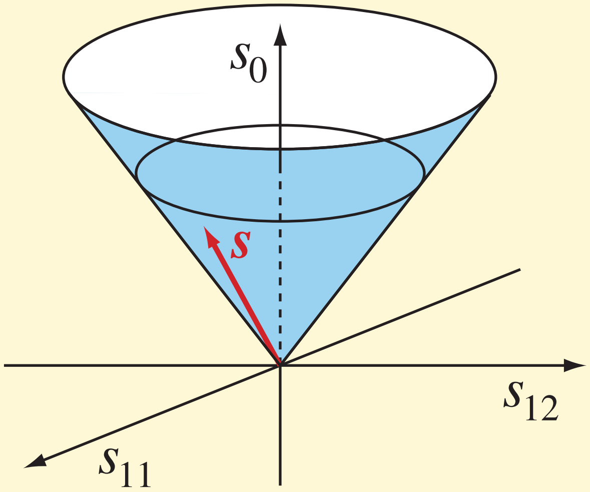
equilibrium analysis w/ von Mises criterion

- incremental problem
- optimization-based approach — potential energy minimization
  - SOCP (second-order cone programming):

    [Bisbos, Makrodimopoulos, & Pardalos '05], [Yonekura & K. '12]

Minimize	(convex quad. fcn.)
subject to	(second-order cones) & (linear eqs.)

- can be solved with a primal-dual interior-point method



$$s_0 \geq \|(s_{11}, s_{12})\|_2$$

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Minimize	(convex quad. fcn.)
subject to	(second-order cones) & (linear eqs.)

    - can be solved with a primal-dual interior-point method
  - equivalent formulation:

    Minimize (nonsmooth convex fcn.)

    - unconstrained optimization
    - suitable for an accelerated proximal gradient method
      - (potentially) efficient for large-scale problems

basic assumptions

- small deformation
  - strain decomposition $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}}$
- strain hardening
  - linear isotropic
  - linear kinematic
- incremental problem

a formulation of incremental problem

- convex, nonsmooth, unconstrained optimization:

$$\text{Min.} \quad \underbrace{\sum_{i=1}^{m} \frac{1}{2} \boldsymbol{\varepsilon}_i^{\mathrm{e}} : \mathbf{C}_i : \boldsymbol{\varepsilon}_i^{\mathrm{e}}}_{\text{elastic energy}} + \underbrace{\sum_{i=1}^{m} \left(\sqrt{\tfrac{2}{3}} R_i \|\boldsymbol{\varepsilon}_i^{\mathrm{p}}\|_{\mathrm{F}} + \frac{1}{3} \|\boldsymbol{\varepsilon}_i^{\mathrm{p}}\|_{\mathrm{F}}^2 \right)}_{\text{plastic dissipation}} - \boldsymbol{f}^{\top} \boldsymbol{u} \qquad (\spadesuit)$$

  - variables: $\boldsymbol{u}$ (inc. disp.), $\boldsymbol{\varepsilon}_i^{\mathrm{p}}$ (inc. plastic strain)
  - $\boldsymbol{\varepsilon}_i^{\mathrm{e}}$ (inc. elastic strain) can be eliminated.
    - substitute $\boldsymbol{\varepsilon}_i^{\mathrm{e}} = B_i \cdot u - \boldsymbol{\varepsilon}_i^{\mathrm{p}}$

a formulation of incremental problem

- convex, nonsmooth, unconstrained optimization:

$$\text{Min.} \quad \underbrace{\sum_{i=1}^{m} \frac{1}{2} \boldsymbol{\varepsilon}_i^{\mathrm{e}} : \mathbf{C}_i : \boldsymbol{\varepsilon}_i^{\mathrm{e}}}_{\text{elastic energy}} + \underbrace{\sum_{i=1}^{m} \left(\sqrt{\tfrac{2}{3}} R_i \|\boldsymbol{\varepsilon}_i^{\mathrm{p}}\|_{\mathrm{F}} + \frac{1}{3} \|\boldsymbol{\varepsilon}_i^{\mathrm{p}}\|_{\mathrm{F}}^2 \right)}_{\text{plastic dissipation}} - \boldsymbol{f}^{\top} \boldsymbol{u} \qquad (\spadesuit)$$

  - variables: $\boldsymbol{u}$ (inc. disp.), $\boldsymbol{\varepsilon}_i^{\mathrm{p}}$ (inc. plastic strain) $\quad \boldsymbol{\varepsilon}_i^{\mathrm{e}} := \boldsymbol{\varepsilon}_i^{\mathrm{e}}(\boldsymbol{u}, \boldsymbol{\varepsilon}_i^{\mathrm{p}})$

- apply proximal gradient method to:

$$(\spadesuit) \quad \Leftrightarrow \quad \text{Min. } f(\boldsymbol{u}, \boldsymbol{\varepsilon}^{\mathrm{p}}) + g(\boldsymbol{\varepsilon}^{\mathrm{p}}) \qquad \text{w/ } g(\boldsymbol{\varepsilon}^{\mathrm{p}}) = \sum_{i=1}^{m} \sqrt{\tfrac{2}{3}} R_i \|\boldsymbol{\varepsilon}_i^{\mathrm{p}}\|_{\mathrm{F}}$$

  - f : convex quadratic $\quad g$: convex, nonsmooth, & simple
  - $\simeq$ group LASSO:

$$\text{Min.} \quad \frac{1}{2} \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \gamma \sum_{l=1}^{m} \|\boldsymbol{x}_l\|_2$$

sketch of algorithm

- iteration of prox. grad. meth.:

$$\begin{bmatrix} \boldsymbol{u}^{(k+1)} \\ \boldsymbol{p} \end{bmatrix} := \begin{bmatrix} \boldsymbol{u}^{(k)} \\ \boldsymbol{\varepsilon}^{\mathrm{p}(k)} \end{bmatrix} - \alpha \nabla^2 f(\boldsymbol{u}, \boldsymbol{\varepsilon}) \begin{bmatrix} \boldsymbol{u}^{(k)} \\ \boldsymbol{\varepsilon}^{\mathrm{p}(k)} \end{bmatrix}$$

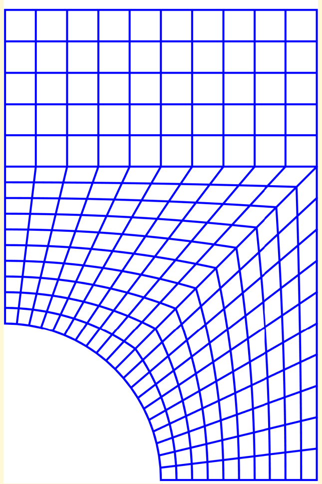
$$\boldsymbol{\varepsilon}_i^{\mathrm{p}(k+1)} := \begin{cases} \boldsymbol{0} & \text{if } \|\boldsymbol{p}_i\| \le \alpha \sqrt{\frac{2}{3}} R_i \\ \left(\|\boldsymbol{p}_i\| - \alpha \sqrt{\frac{2}{3}} R_i \right) \dfrac{\boldsymbol{p}_i}{\|\boldsymbol{p}_i\|} & \text{otherwise} \end{cases} \qquad (\forall i)$$

  - $\nabla^2 f(\boldsymbol{u}, \boldsymbol{\varepsilon})$: a sparse constant matrix
  - $\boldsymbol{p}$: intermediate variable (for notational simplicity)

- very cheap computation per iteration
- only small modification to achieve acceleration

preliminary numerical experiments

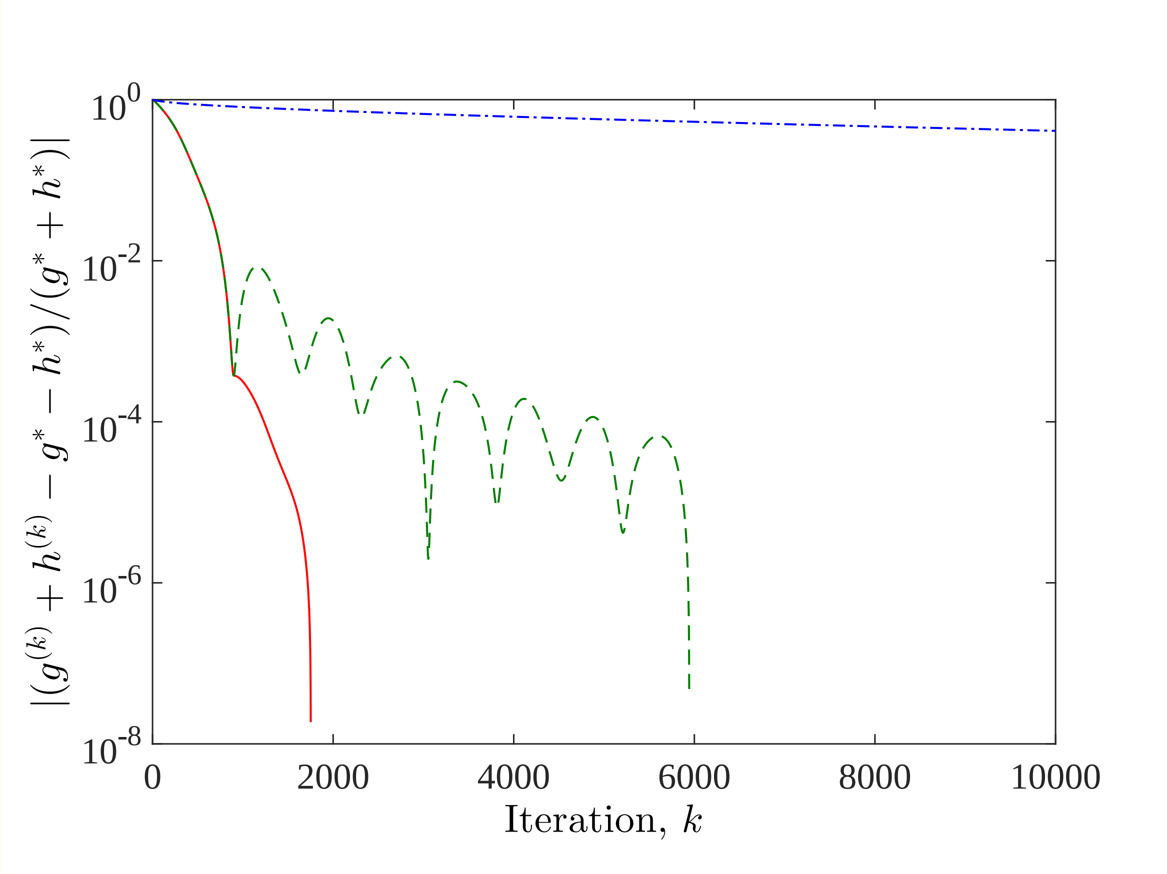
- perforated plate in 3D



- comparison
  - primal-dual interior-point method [Tütüncü, Toh, & Todd '09]
    for second-order cone programming (SOCP): SDPT3 (ver. 4)
    - solves standard form of SOCP.
  - proposed method
    - solves unconstrained nonsmooth convex opt.
    - Matlab implementation

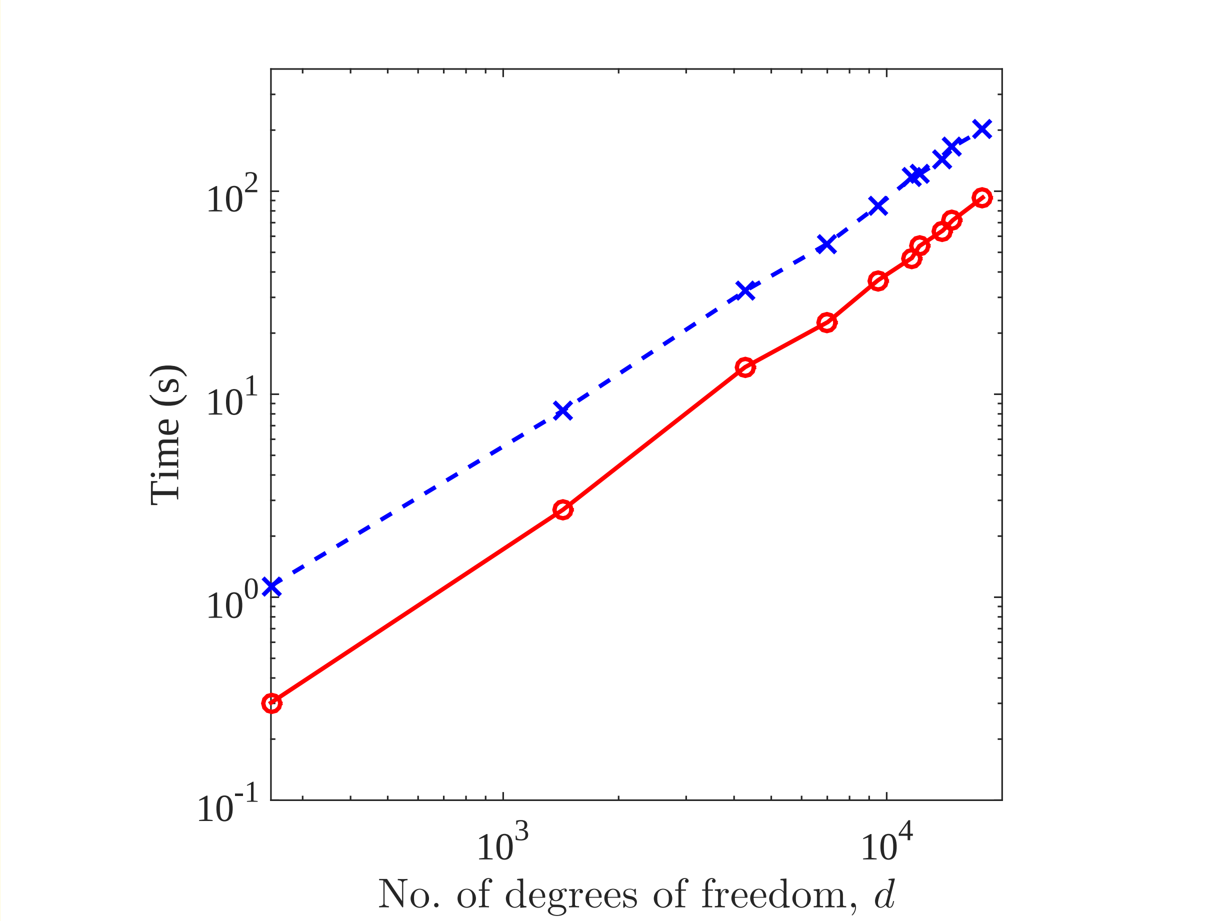
ex.) convergence history

- proximal gradient methods



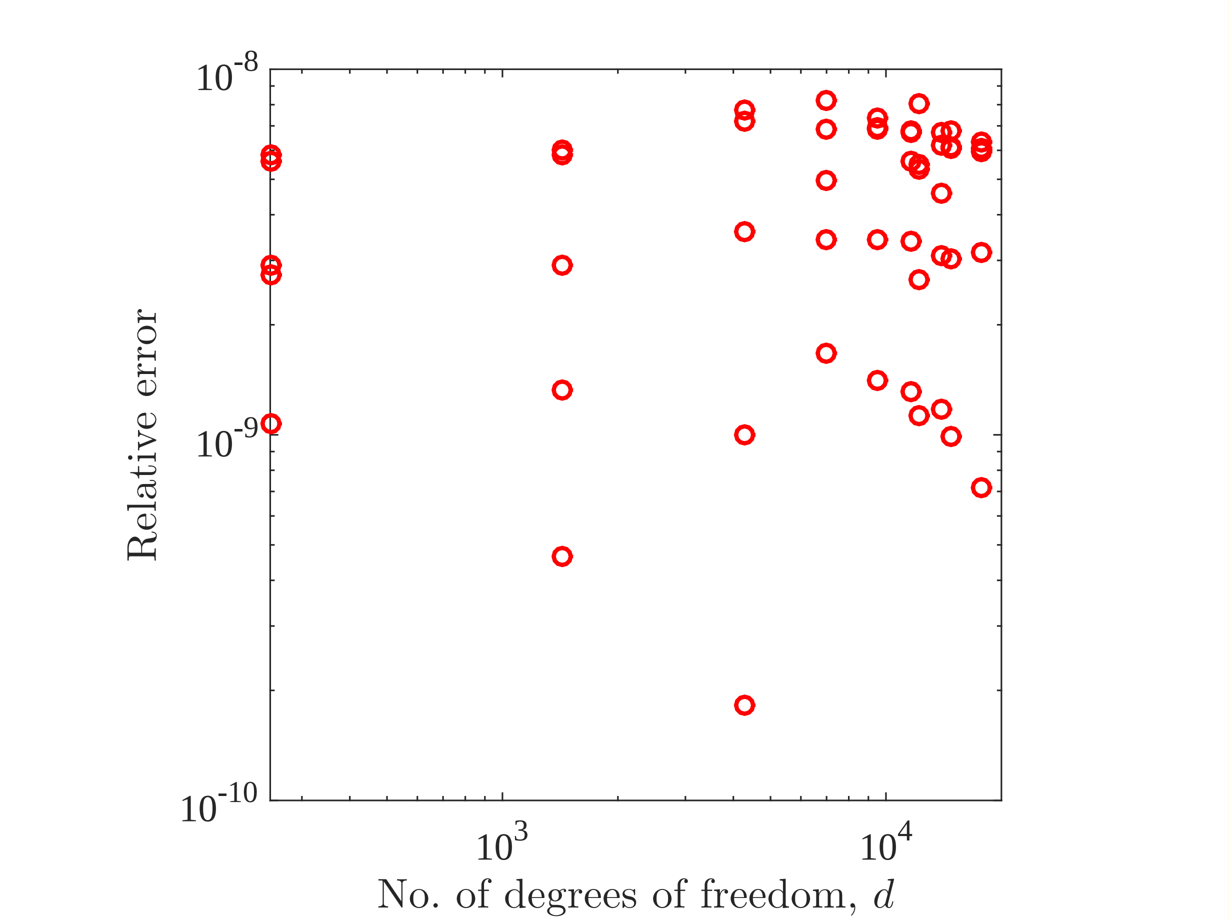
- "- - -" unaccelerated
- "……" accelerated, w/o restart
- "——" accelerated, w/ restart

ex.) computational time



- time vs. DOF
- "×" SDPT3 (interior-point method)
- "○" proposed method

ex.) computational accuracy



- objective value
  - (APGM: proposed) < (SDPT3)

- relative difference = $\dfrac{\text{(SDPT3)} - \text{(APGM)}}{\text{(SDPT3)}}$

conclusions

- incremental elastoplastic analysis w/ von Mises criterion
  - 2nd-order cone prog. (SOCP) $+$ intr.-point meth. (IPM) (exist)
  - unconstrained nonsmooth convex optimization (new)
    - (convex quadratic fcn.) $+$ (sum of ℓ_2-norms)
    - $\simeq$ group LASSO (a regularized least squares)

- accelerated proximal gradient method
  - fast convergence: $O(1/k^2)$
  - cheap computational cost (no system of linear eqs.)
    - fut. wrk.: interpret update & acceleration schemes
  - faster than a standard IPM
    - fut. wrk.: comparison with, e.g., a return-mapping meth.