A Semidefinite Programming Approach to Static Shakedown Analysis with von Mises Yield Criterion and Ellipsoidal Load Domain

Takashi Yamaguchi [†] Yoshihiro Kanno [‡]

[†] University of Tokyo
[‡] Tokyo Institute of Technology

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theme

- SA can be viewed as RO.
  - shakedown analysis (SA)
  - robust optimization (RO)
  - convex optimization
    - second-order cone programming (SOCP)
    - semidefinite programming (SDP)

- Yamaguchi & K. “Ellipsoidal load-domain shakedown analysis with von Mises yield criterion: a robust optimization approach.” *Int. J. Numer. Methods Eng.*, to appear.

second-order cone programming (SOCP)

- linear programming (LP):

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{a}_i^\top \boldsymbol{x} + b_i \geq 0 \quad (i = 1, \dots, m) \end{array}$$

- SOCP:

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{a}_i^\top \boldsymbol{x} + b_i \geq \|P_i \boldsymbol{x} + \boldsymbol{q}_i\| \quad (i = 1, \dots, m) \end{array}$$

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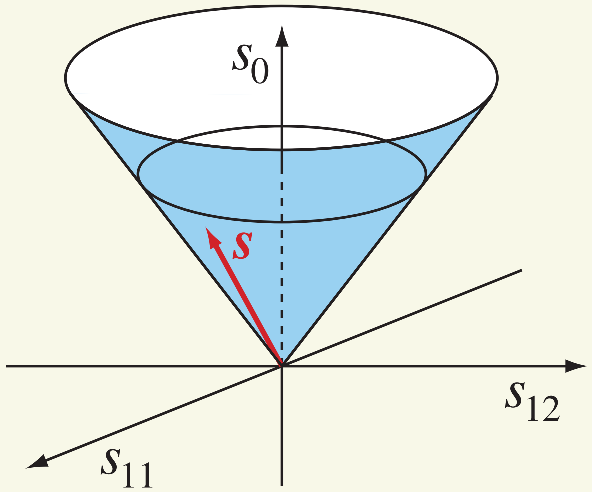
  - a generalization of LP
  - SOC:



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  - convex optim.
    - including LP, QP, etc.
      - LP: $P_i = O$
  - solvable with a primal-dual interior-point method
  - many applications in computational plasticity

semidefinite programming (SDP)

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{m} c_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{m} x_i A_i + B \succeq O \quad : \text{positive semidefinite} \qquad (\diamondsuit) \end{aligned}$$

- $A_1, \dots, A_m, B$: constant symmetric matrices
- ($\diamondsuit$) $\Leftrightarrow$ all eigenvalues ≥ 0
- convex optim.
  - including SOCP, LP, QP, etc.
    - LP: $A_1, \dots, A_m, B$ are diagonal
- solvable with a primal-dual interior-point method

semidefinite programming (SDP)

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- many applications
  - system & control theory (linear matrix inequality)
  - machine learning
  - quantum chemistry
  - relaxations of combinatorial optim.
  - struct. optim.

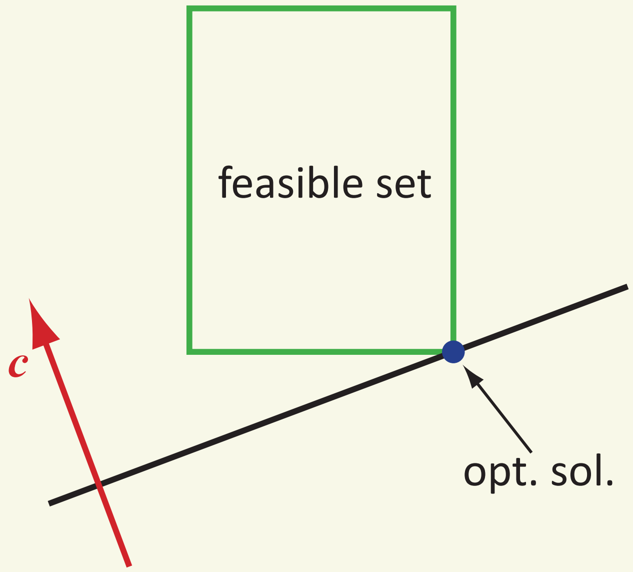
SOCP in plasticity

- LA, SA (w/ a polyhedral load-domain), incremental analysis
  - von Mises
    [Bisbos, Makrodimopoulos, & Pardalos '05], [Yonekura & K. '12]
  - Drucker–Prager, Ilyushin [Makrodimopoulos '06]
  - modified Cam-clay model
    [Makrodimopoulos & Martin '07], [Krabbenhøft & Lyamin '12]
  - Gurson's model for a porus material [Trillat & Pastor '05]
  - 2D Mohr–Coulomb (in plane strain), Nielsen (for a plate)
    [Makrodimopoulos & Martin '06, '07]
  - 2D Mohr–Coulomb (in axisymmetric state) [Tang, Toh, & Phoon '14]
  - Bingham, Herschel–Bulkley yield stress fluids
    [Bleyer, Maillard, de Buhan, & Coussot '15]

SDP in plasticity

- LA, SA (w/ a polyhedral load-domain), incremental analysis
  - Tresca [Bisbos '07], [Bisbos & Pardalos '07]
  - Mohr–Coulomb [Bisbos '07], [Bisbos & Pardalos '07] [Krabbenhøft, Lyamin, & Sloan '07, '08], [Martin & Makrodimopoulos '08]

robust optimization (RO)



- nominal (i.e., conventional) optim.:

$$\begin{aligned} &\text{Min.} && \boldsymbol{c}^\top \boldsymbol{x} \\ &\text{s.t.} && A\boldsymbol{x} \le \boldsymbol{b} \end{aligned}$$

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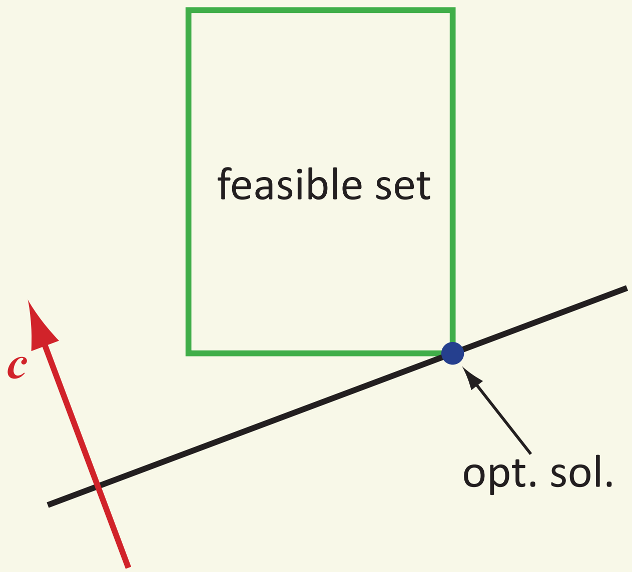

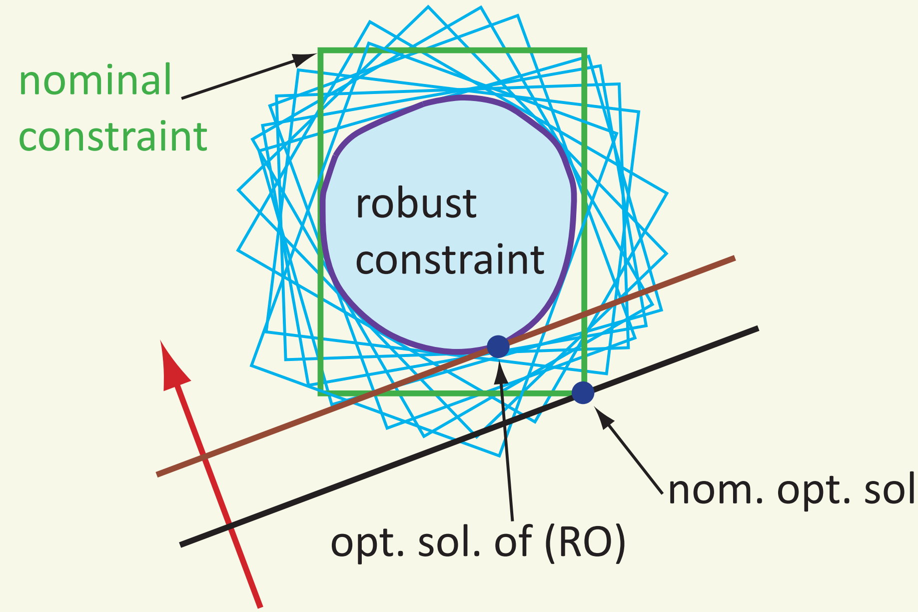

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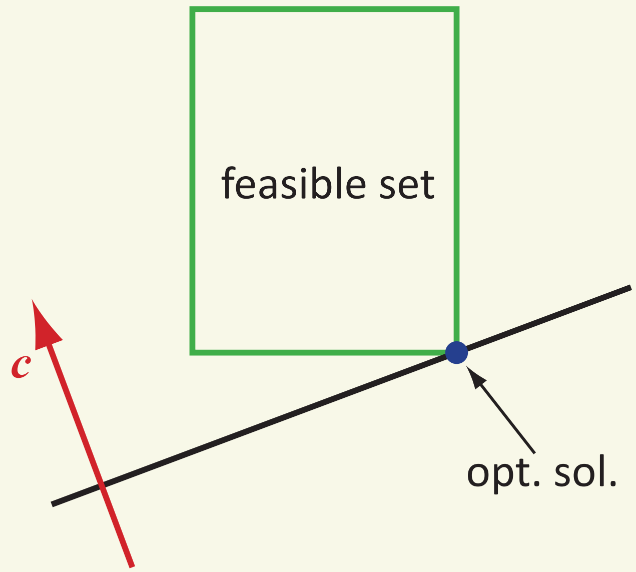
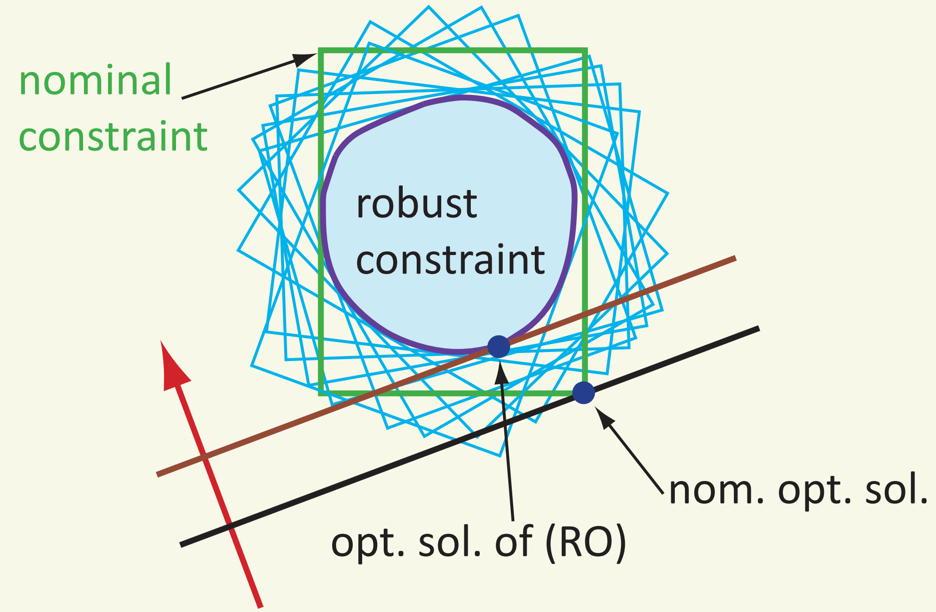
$$\begin{array}{ll} \text{Min.} & \boldsymbol{c}^{\top}\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \le \boldsymbol{b} \end{array}$$

- robust optim.:

$$\begin{array}{ll} \text{Min.} & \boldsymbol{c}^{\top}\boldsymbol{x} \\ \text{s.t.} & A(z)\boldsymbol{x} \le \boldsymbol{b}(z) \quad (\forall z \in \mathcal{U}) \end{array} \qquad \text{(RO)}$$

robust optimization





- "feasible set of L.H.S. is convex"
  $\Rightarrow$ "feasible set of R.H.S. is convex"
  - $\because$ The intersection of convex sets is convex.

- In some cases, infinitely many constraints on R.H.S. can be reduced to finitely many constraints.
  - The difficulty of (RO) depends on the uncertainty model and the nominal constraint form.

robust constraint

- nominal (i.e., conventional) constraint:

$$g_i(\boldsymbol{x}) \leq 0$$

- uncertainty in the constraint fnctn.:

$$g_i(\boldsymbol{x}, z), \quad z \in \mathcal{U}$$

  - z : uncertain parameter
  - $\mathcal{U}$: set of z

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- robust cstr.:

$$g_i(\boldsymbol{x}, z) \le 0 \quad (\forall z \in \mathcal{U})$$

$$\Updownarrow$$

$$\max\{g_i(\boldsymbol{x}, z) \mid z \in \mathcal{U}\} \le 0$$

: cstr. in the worst-case scenario

shakedown analysis (SA) and robust optimization (RO)

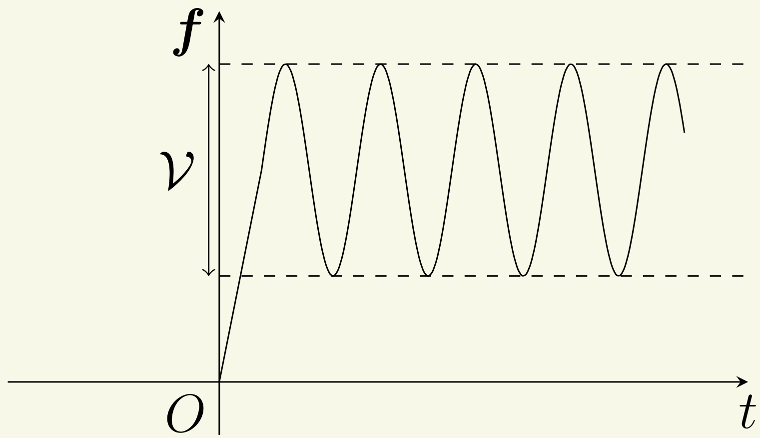
- a viewpoint:
  - LA is optim.
  - SA is robust optim.

shakedown analysis (SA) and robust optimization (RO)

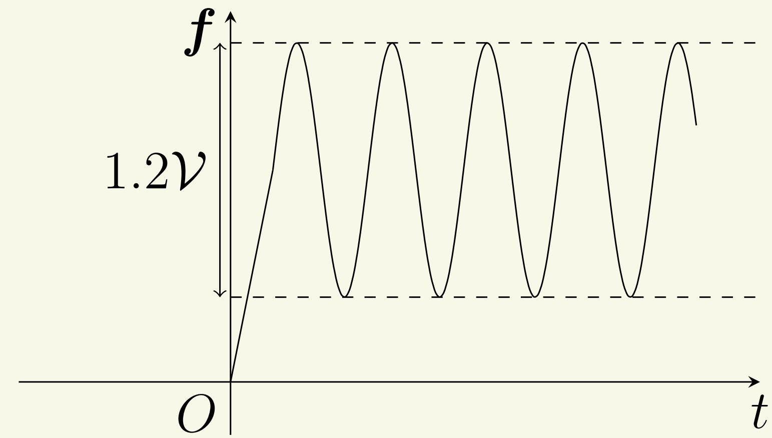
- a viewpoint:
  - LA is optim.
  - SA is robust optim.
- load $\lambda \boldsymbol{p} + \boldsymbol{q}$ (λ : load multiplier)
  - LA
    - $\boldsymbol{p}$ and $\boldsymbol{q}$ are data (const.).
    - Find maximum λ such that the structure can sustain.
  - SA
    - $\boldsymbol{p}$ is uncertain (varies).
      - All $\boldsymbol{p} \in \mathcal{V}$ with given $\mathcal{V}$ are considered.
    - Find maximum λ such that the structural response can converge to purely elastic one.

illustrative ex. of shakedown

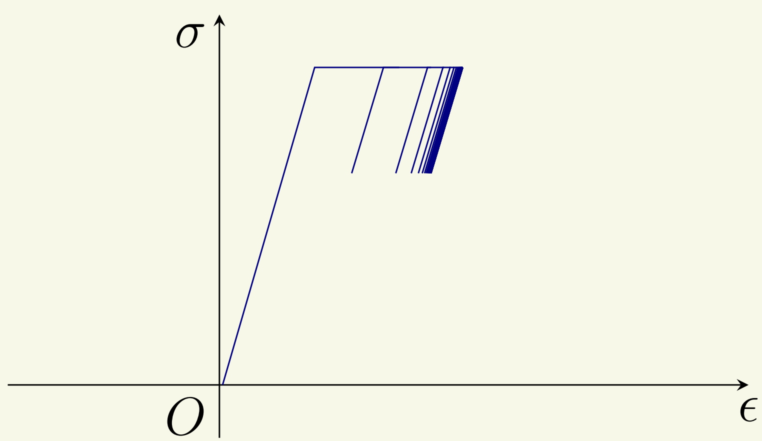
- load: $\lambda \boldsymbol{p} + \boldsymbol{q}$ with $\boldsymbol{p} \in \mathcal{V}$



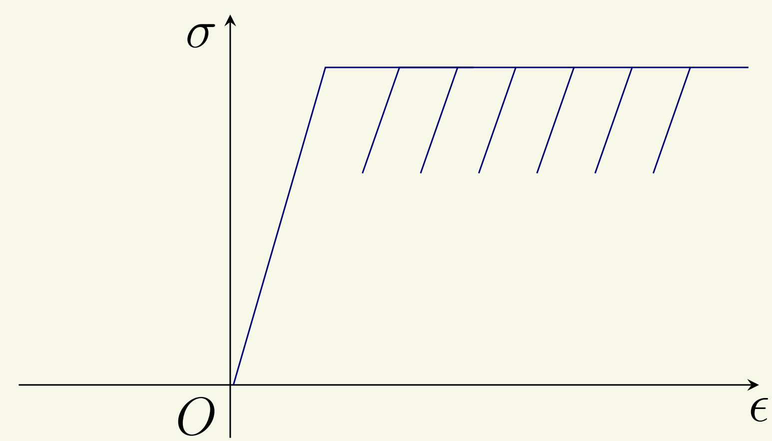
ex.) $\lambda = 1.0$



ex.) $\lambda = 1.2$



shakedown

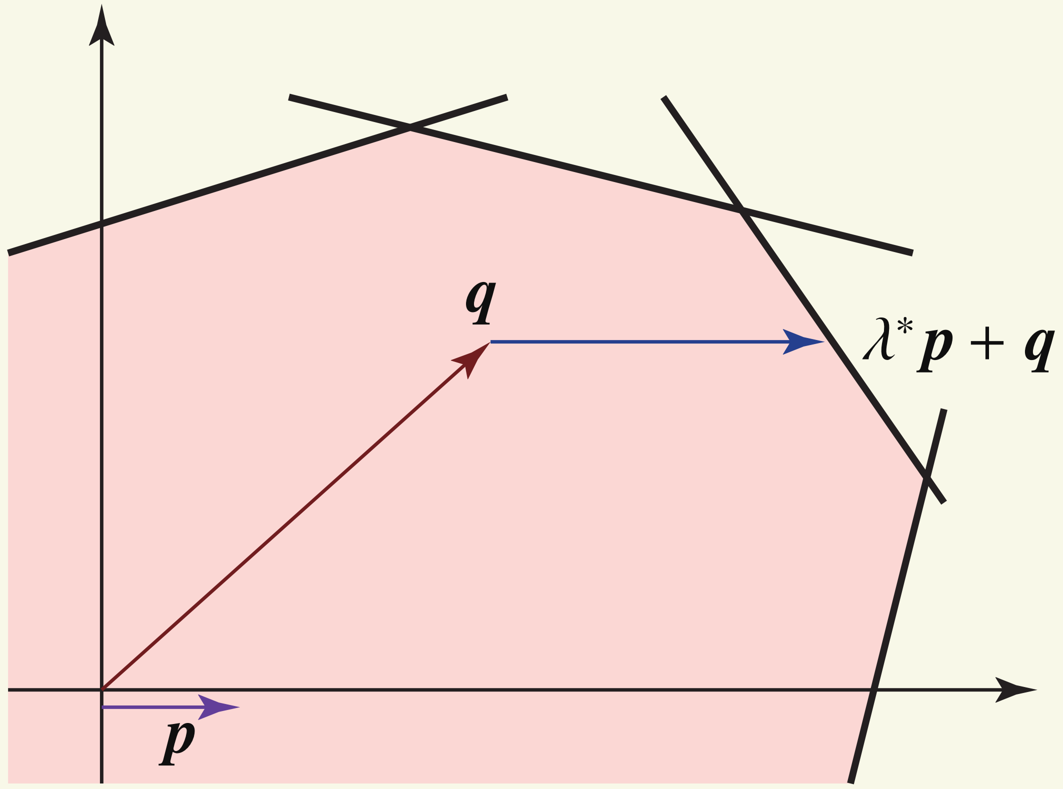


not shakedown

- In this ex., "$1.0 \leq$ shakedown factor < 1.2".

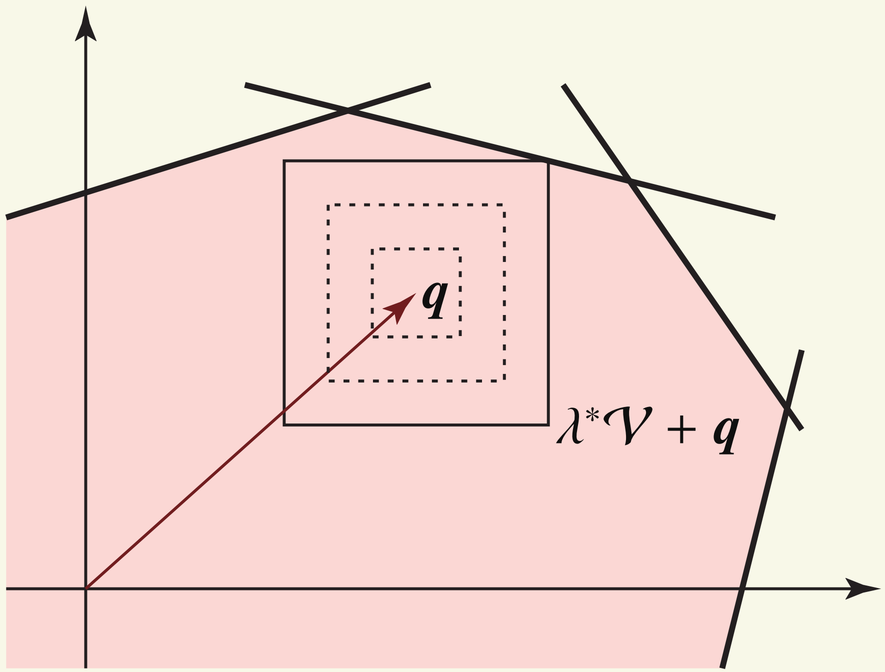
intuitive understanding

- LA
  - load $\lambda \boldsymbol{p} + \boldsymbol{q}$ (w/ constant $\boldsymbol{p}$ and $\boldsymbol{q}$)
  - maximize the magnitude of a load lying inside the yield surface



intuitive understanding

- SA
  - load $\lambda \boldsymbol{p} + \boldsymbol{q}$ (w/ any $\boldsymbol{p} \in \mathcal{V}$)
  - maximize the size of a load domain lying inside the yield surface
    - ex.) Suppose that the load domain $\mathcal{V}$ is a square centered at $\boldsymbol{q}$.



lower-bound theorems

- LA

$$
\begin{aligned}
\text{Max.} \quad & \lambda \\
\text{s. t.} \quad & \sum_{l=1}^{r} H_l \boldsymbol{\sigma}_l = \lambda \boldsymbol{p} + \boldsymbol{q} && \text{(force-balance eq.)} \\
& f(\boldsymbol{\sigma}_l) \leq 0 && \text{(yield cond.)}
\end{aligned}
$$

  - $\boldsymbol{\sigma}_l$: stress

lower-bound theorems

- LA

$$\begin{aligned} \text{Max.} \quad & \lambda \\ \text{s. t.} \quad & \sum_{l=1}^{r} H_l \boldsymbol{\sigma}_l = \lambda \boldsymbol{p} + \boldsymbol{q} && \text{(force-balance eq.)} \\ & f(\boldsymbol{\sigma}_l) \le 0 && \text{(yield cond.)} \end{aligned}$$

- SA

$$\begin{aligned} \text{Max.} \quad & \lambda \\ \text{s. t.} \quad & \sum_{l=1}^{r} H_l \boldsymbol{r}_l = \boldsymbol{0} && \text{(self-equilibrium)} \\ & f(\hat{\boldsymbol{\sigma}}_l(\lambda \boldsymbol{p} + \boldsymbol{q}) + \boldsymbol{r}_l) \le 0 \quad (\forall \boldsymbol{p} \in \mathcal{V}) && \text{(yield cond.)} \end{aligned}$$

  - $\boldsymbol{\sigma}_l$: stress, $\boldsymbol{r}_l$: residual stress
  - $\hat{\boldsymbol{\sigma}}_l(\boldsymbol{q})$: elastic stress due to load $\boldsymbol{q}$

lower-bound theorems

- LA $\leftrightarrow$ optimization

$$\begin{aligned} \text{Max.} \quad & \lambda \\ \text{s.t.} \quad & \sum_{l=1}^{r} H_l \boldsymbol{\sigma}_l = \lambda \boldsymbol{p} + \boldsymbol{q} && \text{(force-balance eq.)} \\ & f(\boldsymbol{\sigma}_l) \le 0 && \text{(yield cond.)} \end{aligned}$$

- SA $\leftrightarrow$ robust optimization

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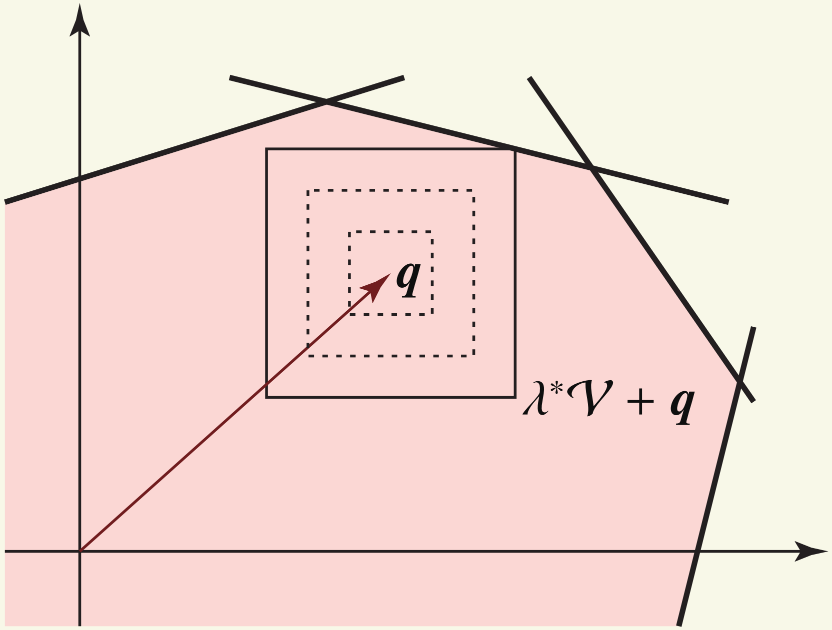
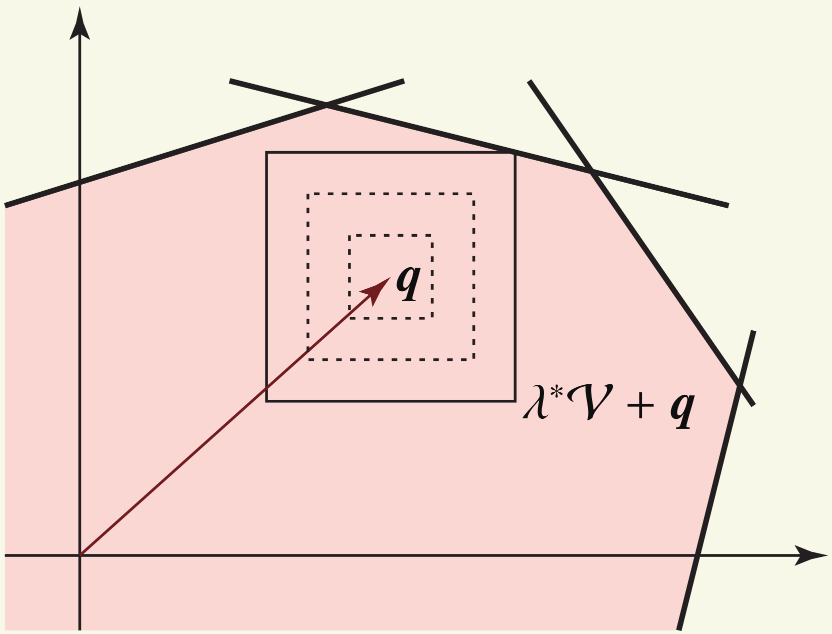
yield criterion, load set, & SA

- Difficulty of SA depends on YC and load set $\mathcal{V}$.
- piecewise-linear YC
  - SA — max. of load multiplier λ



yield criterion, load set, & SA

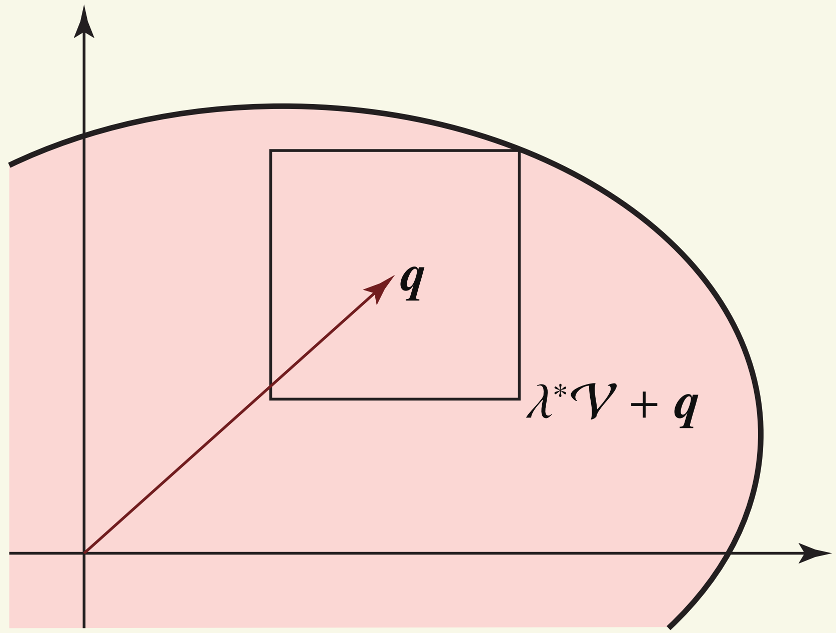
- Difficulty of SA depends on YC and load set $\mathcal{V}$.
- piecewise-linear YC
  - $\mathcal{V}$ is a polytope $\Rightarrow$ SA = linear programming (LP)



  - All vertices of $\mathcal{V}$ should lie inside the yield surface.
  - Enumerating YCs at all vertices results in LP. [Maier '69], etc.

yield criterion, load set, & SA

- Difficulty of SA depends on YC and load set $\mathcal{V}$.
- nonlinear convex YC
  - $\mathcal{V}$ is a polytope $\Rightarrow$ SA = convex optimization



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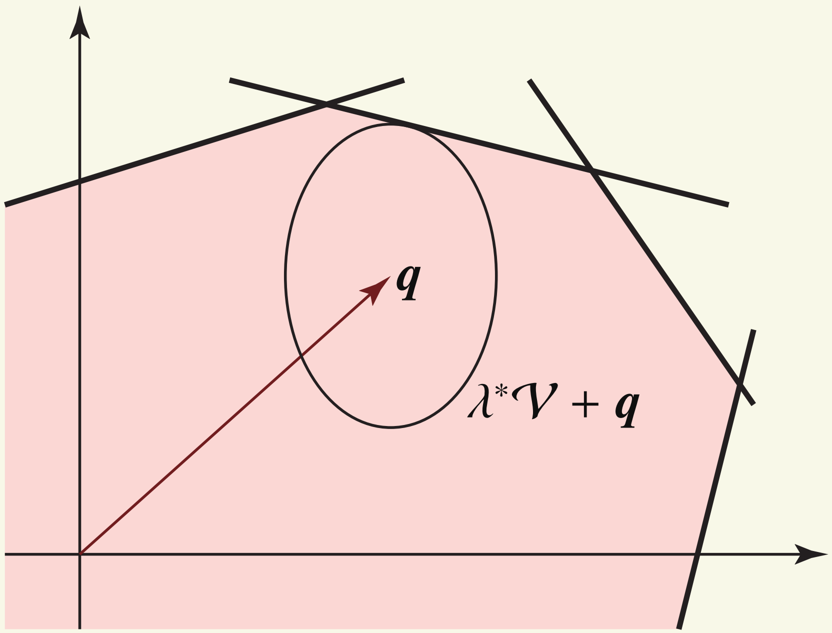
  - Enumerating YCs at all vertices results in convex optimization.

[Bisbos, Makrodimopoulos, & Pardalos '05]

[Makrodimopoulos '06], [Bisbos '07]

yield criterion, load set, & SA

- Difficulty of SA depends on YC and load set $\mathcal{V}$.
- piecewise-linear YC
  - $\mathcal{V}$ is an ellipsoid.



- Infinitely many vertices!
  - SA is "robust LP".
- But, SA can be reduced to LP. [Bisbos & Ampatzis '08]

[Bisbos & Ampatzis '08]

- constraint for SA (linear yield criterion):

$$\boldsymbol{a}_i^\top (\lambda \boldsymbol{p} + \boldsymbol{r}_l) \le b_i \quad (\forall \boldsymbol{p} \in \mathcal{V}) \qquad (\diamondsuit)$$

← robust linear inequality constraint

- λ : load multiplier
- $\boldsymbol{p}$: variable load
- $\boldsymbol{r}_l$: residual stress
- $\boldsymbol{a}_i, b_i$: const.

[Bisbos & Ampatzis '08]

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- ellipsoidal variable-load set:

$$\mathcal{V} = \{Q\boldsymbol{z} \mid \|\boldsymbol{z}\| \le 1\}$$

$\rightarrow$ use a technique in RO

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- ellipsoidal variable-load set:

$$\mathcal{V} = \{Q\boldsymbol{z} \mid \|\boldsymbol{z}\| \le 1\}$$

- robust constraint:

$$(\diamondsuit) \quad \Leftrightarrow \quad \max\{\boldsymbol{a}_i^\top(\lambda Q\boldsymbol{z} + \boldsymbol{r}_l) \mid \|\boldsymbol{z}\| \le 1\} \le b_i \qquad (\spadesuit)$$

- elimination of $\boldsymbol{z}$:

$$\text{L.H.S. of } (\spadesuit) = \lambda \max\{\boldsymbol{z}^\top(Q^\top \boldsymbol{a}_i) \mid \|\boldsymbol{z}\| \le 1\} + \boldsymbol{a}_i^\top \boldsymbol{r}_l$$
$$= \lambda \|Q^\top \boldsymbol{a}_i\| + \boldsymbol{a}_i^\top \boldsymbol{r}_l$$

[Bisbos & Ampatzis '08]

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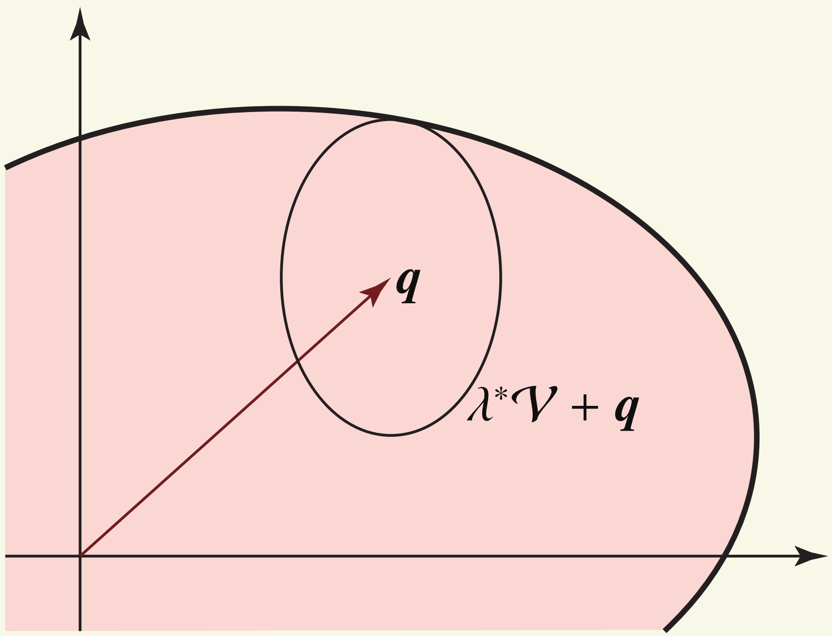
result (lin. ineq. w.r.t. λ & $\boldsymbol{r}_l$):

$$(\diamondsuit) \quad \Leftrightarrow \quad \lambda \|Q^\top \boldsymbol{a}_i\| + \boldsymbol{a}_i^\top \boldsymbol{r}_l \le b_i$$

our result

- von Mises yield criterion
  - $\mathcal{V}$ is an ellipsoid.



- "curved yield criterion" & "curved load set"
  - Infinitely many vertices!
  - SA is "robust SOCP".
- SA can be reduced to SDP. [Yamaguchi & K. '16]

robust second-order cone constraint

- a theorem in RO:

$$\text{(a)}\ \overbrace{\|A\boldsymbol{x} + \boldsymbol{b} + C\boldsymbol{z}\| \le d}^{(\clubsuit)} \quad \overbrace{(\forall \boldsymbol{z} : \|\boldsymbol{z}\| \le 1)}^{(\heartsuit)}$$

$$\Updownarrow$$

$$\text{(b)}\ \exists \tau \in \mathbb{R} : \begin{bmatrix} I & C & A\boldsymbol{x} + b \\ & \tau I & O \\ \text{(symm.)} & & d^2 - \tau \end{bmatrix} \text{ is p.s.d.}$$

Ben-Tal, El Ghaoui, Nemirovski: *Robust Optimization* (2009).

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- The theorem can be shown by using "S-lemma":

$$\text{(a)}\ \overbrace{\boldsymbol{y}^\top U \boldsymbol{y} \ge 0}^{(\heartsuit)} \Rightarrow \overbrace{\boldsymbol{y}^\top V \boldsymbol{y} \ge 0}^{(\clubsuit)}$$

$$\Updownarrow$$

$$\text{(b)}\ \exists w \ge 0 : V - wU \text{ is p.s.d.}$$

  - asm.: U and V are symmetric matrices, and U has at least one positive eigenvalue.

robust second-order cone constraint

- a theorem in RO:

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- In SA,
  - ($\clubsuit$) $\leftrightarrow$ yield criterion (von Mises)
  - $\boldsymbol{x} \leftrightarrow$ residual stress
  - $C\boldsymbol{z}$ with ($\heartsuit$) $\leftrightarrow$ $\lambda \boldsymbol{p}$ with "$\boldsymbol{p} \in$ (an ellipsoid)"

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- In SA,
  - ($\clubsuit$) $\leftrightarrow$ yield criterion (von Mises)
  - $\boldsymbol{x}$ $\leftrightarrow$ residual stress
  - $C\boldsymbol{z}$ with ($\heartsuit$) $\leftrightarrow$ $\lambda \boldsymbol{p}$ with "$\boldsymbol{p} \in$ (an ellipsoid)"
- Consequently, SA can be reduced to SDP.

  $\rightarrow$ solvable with an interior-point method

SDP for SA

- SA w/ infinitely many constraints (robust SOCP):

$$\begin{aligned} \text{Max.} \quad & \lambda \\ \text{s.t.} \quad & \sum_{l=1}^{r} H_l \boldsymbol{r}_l = \boldsymbol{0}, \\ & \|T(S_l(\lambda \boldsymbol{p} + \boldsymbol{q}) + \boldsymbol{r}_l)\| \le R_l \ (\forall \boldsymbol{p} \in \{F\boldsymbol{\theta} : \|\boldsymbol{\theta}\| \le 1\}), \quad l = 1, \dots, r. \end{aligned}$$

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- SDP formulation:

$$\begin{aligned} \text{Max.} \quad & \lambda \\ \text{s.t.} \quad & \sum_{l=1}^{r} H_l \boldsymbol{r}_l = \boldsymbol{0}, \\ & \begin{bmatrix} I & \lambda T S_l F & T(\boldsymbol{r}_l + S_l \boldsymbol{q}) \\ & \tau_l I & O \\ \text{(symm.)} & & R_l^2 - \tau_l \end{bmatrix} \succeq O, \quad l = 1, \dots, r. \end{aligned}$$

  - variables: λ, $\boldsymbol{r}_l$, τ_l

motivations of ellipsoidal load-set (1/2)

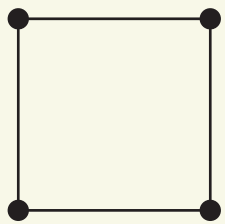
- link w/ probability distribution
  - less informative than a full prob. distr.
  - For the load vector, assume:
    - $\boldsymbol{q}$: mean vector
    - Σ : covariance matrix

    then,
    - $\lambda \boldsymbol{p} + \boldsymbol{q}$ with $\boldsymbol{p} \in \{\Sigma^{1/2}\boldsymbol{\theta} \mid \|\boldsymbol{\theta}\| \le 1\}$ can be a load model.

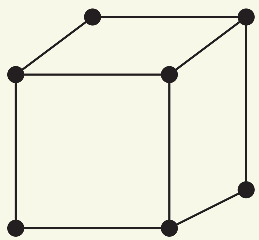
    Indeed,
    - if the load has the multivaluated normal distr., probability is given by $f_d(\lambda^2)$.
      - $f_d(\lambda^2)$: density fnctn. of χ^2-distr. with d DOF.

motivations of ellipsoidal load-set (2/2)

- box load-set $\mathcal{V}_{\rm box} = \{F\boldsymbol{\theta} \mid \|\boldsymbol{\theta}\|_\infty \le 1\}$ [Simon & Weichert '12], etc.
- vertices of $\mathcal{V}_{\rm box}$



2^2 vertices



2^3 vertices

k-dim.

$\downarrow$

2^k vertices

- The von Mises yield criterion is SOCP representable.
  - Enumerating YCs at all vertices of $\mathcal{V}_{\rm box}$ results in SOCP for SA.
  - (# of SOC constraints) $= 2^k \times$ (# of Gauss points)
    $\leftarrow$ exponentially-large SOCP
- If we approximate $\mathcal{V}_{\rm box}$ by an ellipsoid,
  - SDP for SA has r positive semidefinite cstr. w/ $(k+7)$-dim.
    $\leftarrow$ linearly-increasing w.r.t. k

conclusions

- Melan's (lower bound) shakedown theorem
  - robust optimization
- new result:
  - von Mises yield criterion, ellipsoidal load-domain
    - robust SOCP $\rightarrow$ equivalent SDP
      (intractable) (tractable)
- existing results:
  - piecewise-linear y.c., polyhedral l.d. / ellipsoidal l.-d.
    - robust LP $\rightarrow$ equivalent LP
  - convex y.c., polyhedral l.d.
    - robust convex prog. $\rightarrow$ equivalent convex prog.
    - for von Mises: robust SOCP $\rightarrow$ equivalent SOCP