Large Deflection Analysis of Cable Networks
by Second-Order Cone Program

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Large deformation analysis of cable network:

- **Given**
  - supports: $b_i^0 \in \mathbb{R}^3$
  - external forces: $\bar{f} \in \mathbb{R}^{3N_n}$
  - member unstressed length: $l_i^0$

- **Find**
  - internal nodes: $x \in \mathbb{R}^{3N_n}$
Stress unilateral behavior:

**Backgrounds:**

- transmit **only tension force**

  1. **trial-and-error**
     - (a) assume whether in **tensile** or **slackening state**
     - (b) check the obtained solution
     - (c) correct the assumption
  
  2. **singularity** of the tangent stiffness matrix

**The proposed algorithm:**

1. **Second-Order Cone Programming (SOCP) problem**
   \[ \text{min. of Total Potential Energy} \]

2. **pin-joints**, frictionless joints

3. **no assumption** on stress state

4. convergence
   - (a) **unstable** cable networks
   - (b) estimate of stress states is difficult
Second-Order Cone Programming : SOCP

1. convex programming
2. including LP, QP, etc.
3. included in SDP
4. primal-dual interior-point method
   (Monteiro and Tsuchiya, 2000)
   • polynomial time convergence
5. applications
   (a) truss topology optimization (Jarre et al., 1998)
   (b) magnetic shield design
       (Sasakawa and Tsuchiya, 2000)
   (c) antenna array design (Scholnik and Coleman, 2000)
Second-Order Cone Programming: SOCP

Minimize \[ b^\top y \]

subject to \[ A^\top y + x = c, \]
\[ x_0 \geq \|x_1\|. \]

Variable vectors
\[ x = (x_0, x_1) \in \mathbb{R}^n, \quad y \in \mathbb{R}^m \]

Definitions
\[ \|x_1\| = (x_1^\top x_1)^{1/2} \] : Euclidean norm
\[ \mathcal{K}(n) = \{(x_0, x_1)|x_0 \geq \|x_1\|\} \] : second-order cone

Constant matrix and vectors
\[ A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c \in \mathbb{R}^n \]

Second-order cone in 3-dimensional space
Minimization problem of TPE

\[
\text{TPE} : \text{Minimize} \quad \sum_{i=1}^{N^m} w_i(y_i) - \overline{f}^\top x
\]

subject to \[
w_i(y_i) = \begin{cases} 
\frac{1}{2} k_i y_i^2, & (y_i \geq 0), \\
0, & (y_i < 0), 
\end{cases}
\]
\[
y_i = \| B_i x + b_i^0 \| - l_i^0.
\]

Variables:
\[
y_i : \text{member elongation}, \quad x : \text{internal nodes}
\]

Given:
\[
\overline{f} : \text{external forces}, \quad l_i^0 : \text{member unstressed length}, \\
b_i^0 : \text{supports}, \quad B_i^0 : \text{adjacency matrices}
\]

Strain energy:
\[
w_i(y_i) : \text{strain energy}, \quad k_i : \text{extensional stiffness}
\]
The image shows two graphs:

- **Axial force-elongation** graph:
  - The graph plots axial force ($N_i$) against elongation ($y_i$).
  - There is a linear relationship between $N_i$ and $y_i$ with a slope $k_i$.
  - The origin $(0, 0)$ is labeled, indicating no force or elongation.

- **Strain energy** graph:
  - The graph plots strain energy ($w_i$) against elongation ($y_i$).
  - The graph shows a nonlinear relationship, typical for strain energy, increasing rapidly with elongation.
  - The origin $(0, 0)$ is labeled, indicating no strain energy at zero elongation.

The graphs illustrate the relationship between force and elongation as well as the strain energy associated with this deformation.
1. nonconvex problem

2. $w_i$ depends on the sign of $y_i$
   - assumptions are required
   - trial-and-error process
Minimization problem of TPE:

\[
\text{TPE} : \quad \text{Minimize } \sum_{i=1}^{N^m} w_i(y_i) - \bar{f}^\top x
\]

subject to

\[
w_i(y_i) = \begin{cases} 
\frac{1}{2} k_i y_i^2, & (y_i \geq 0), \\
0, & (y_i < 0),
\end{cases}
\]

\[
y_i = \| B_i x + b^0_i \| - l^0_i.
\]

SOCP formulation:

\[
\text{SOCP} : \quad \text{Minimize } \sum_{i=1}^{N^m} \frac{1}{2} k_i y_i^2 - \bar{f}^\top x
\]

subject to

\[
y_i \geq \| B_i x + b^0_i \| - l^0_i.
\]
in tensile state:

\[ w_i \]

\[ y_i \]

stray energy

\[ 0 \]

\[ \|B_ix + b_i^0\| - l_i^0 \]

TPE

SOCP

optimal

feasible
slackening:

\[ \|B_i x + b_i^0\| - \ell_i^0 \]

\[ w_i \]

\[ y_i \]

TPE

SOCP
SOCP:

1. has the same optimizer as that of TPE
2. convex problem
3. efficient algorithm
   - no assumption is required—no trial-and-error process
   - polynomial-time convergence—IPM
Nonlinear constitutive law $\implies$ SOCP:

Bi-linear material.

quadratic-linear constitutive law.

**stiffness reduction:**

1. small elongation
2. strand cable becomes lose
3. deflection of cable $\Leftarrow$ own weight
Optimality conditions:

SOCP: Minimize \[ \sum_{i=1}^{N^m} \frac{1}{2} k_i y_i^2 - \bar{f}^T x \]
subject to \[ y_i \geq \|B_i x + b_i^0\| - \bar{l}_i. \]

Lagrangian multipliers

\[ q_i \in \mathbb{R}, \quad v_i \in \mathbb{R}^3, \quad (i = 1, 2, \ldots, N^m). \]

KKT conditions:

\[ q_i = k_i y_i, \quad \text{: constitutive law} \]
\[ \sum_{i=1}^{N^m} B_i^T v_i + \bar{f} = 0, \quad \text{: equilibrium equations} \]
\[ (y_i + \bar{l}_i^0)q_i + (B_i x + b_i^0)^T v_i = 0, \]
\[ y_i \geq \|B_i x + b_i^0\| - \bar{l}_i^0, \quad q_i \geq \|v_i\|. \]

- \( q_i \): axial force
- \( v_i \): internal force vector
Examples:

SOCP : solve SOCP problem by IPM.
TPE : solve min. of TPE by IPM.
NR : Newton-Raphson method (tangent stiffness).

Model (I)  

Model (II)  

Model (III)
Self-equilibrium shape analysis

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Model</th>
<th>Steps</th>
<th>CPU Time (sec.)</th>
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<td>Total</td>
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<td>(III) 432</td>
<td>22</td>
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1. CPU time

- $O(n) < \text{SOCP} < \text{TPE} < O(n^2)$
- $O(n^2) < \text{NR} < O(n^3)$

2. # of variables : $n$

- $\text{SOCP} > \text{TPE} > \text{NR}$
## Dependence of initial solutions

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<th>(B)</th>
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1. SOCP (A) \( \cong \) SOCP (B)

2. NR (B) : fail

   tangent stiffness matrix : singular

![Initial solution (A).](image1)

![Initial solution (B).](image2)
Initial configuration with many slackening members

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<tr>
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<td>432</td>
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** : did not converge within 1000 steps.

1. **SOCP** : converge
2. **TPE, NR** : fail (oscillation)
   - trial-and-error process

Model (III'): Slackening members.
### Bi-linear material

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**External loads.**

**Members in plastic state**

![Graph showing load versus displacement](image1)

![Graph showing stress-strain relationship](image2)
Conclusions:

1. An **SOCP formulation** has been proposed for large deformation analysis of cable networks.
   (a) **pin-joints**
   (b) **frictionless joints**
   (c) **nonlinear material**

2. Equilibrium configurations are obtained by solving SOCP problems by using **Interior Point Method**.
   (a) **no assumption** on stress state
   (b) **no process of trial and error**

3. SOCP formulation is **more efficient** than
   (a) min. of **Total Potential Energy**.
   (b) **Newton-Raphson method** based on tangent stiffness.