A Heuristic for Truss Topology Optimization under Constraint on Number of Nodes

Yoshihiro Kanno   Shinnosuke Fujita

(Tokyo Institute of Technology)

September 27, 2017 (IASS 2017)

number of nodes in truss topology optim.

- #node $\leftrightarrow$ fabrication cost of a truss
  - cost of nodes
  - #member $\sim$ #nodes

number of nodes in truss topology optim.

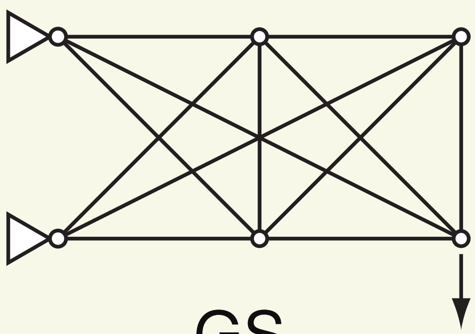
- #node ↔ fabrication cost of a truss
  - cost of nodes
  - #member ∼ #nodes
- optimization incorporating fabrication cost
  - min. weighted sum of struct. vol. & fabn. cost
    - fabn. cost ∼ #member [Asadpoure, Guest, & Valdevit ’15]
    - fabn. cost ∼ #node [Torii, Lopez, & Miguel ’16]

number of nodes in truss topology optim.

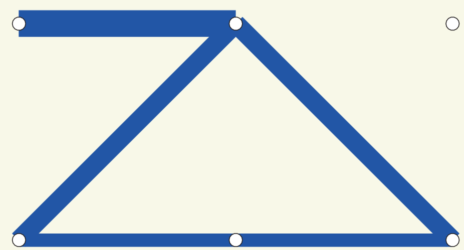
- #node $\leftrightarrow$ fabrication cost of a truss
  - cost of nodes
  - #member $\sim$ #nodes
- optimization incorporating fabrication cost
  - min. weighted sum of struct. vol. & fabn. cost
    - fabn. cost $\sim$ #member [Asadpoure, Guest, & Valdevit '15]
    - fabn. cost $\sim$ #node [Torii, Lopez, & Miguel '16]
- motivation
  - explicit control of #node—combinatorial constraint
  - min. compliance
  - algorithm: alternating direction method of multipliers (ADMM)

revisiting ground structure

- conventional compliance min.:
  - Overlapping bars in GS are omitted *a priori*.



GS

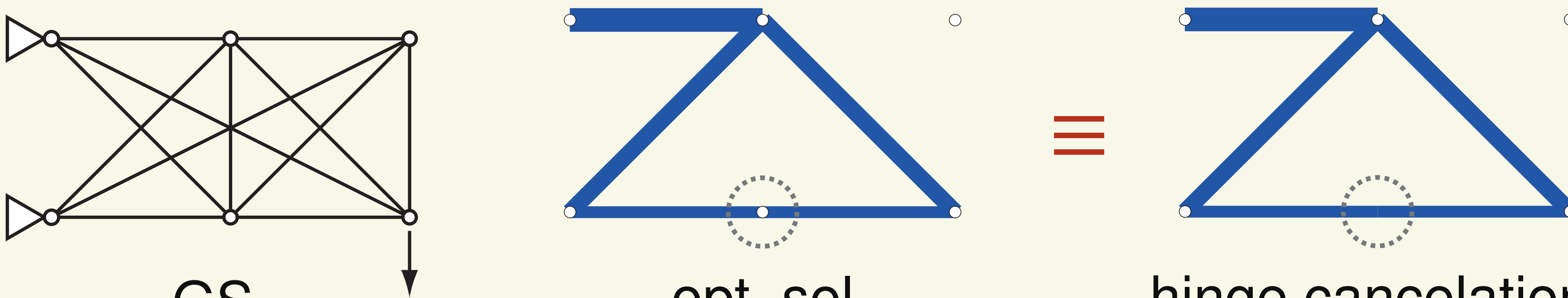


opt. sol.

≡

revisiting ground structure

- conventional compliance min.:
  - Overlapping bars in GS are omitted *a priori*.



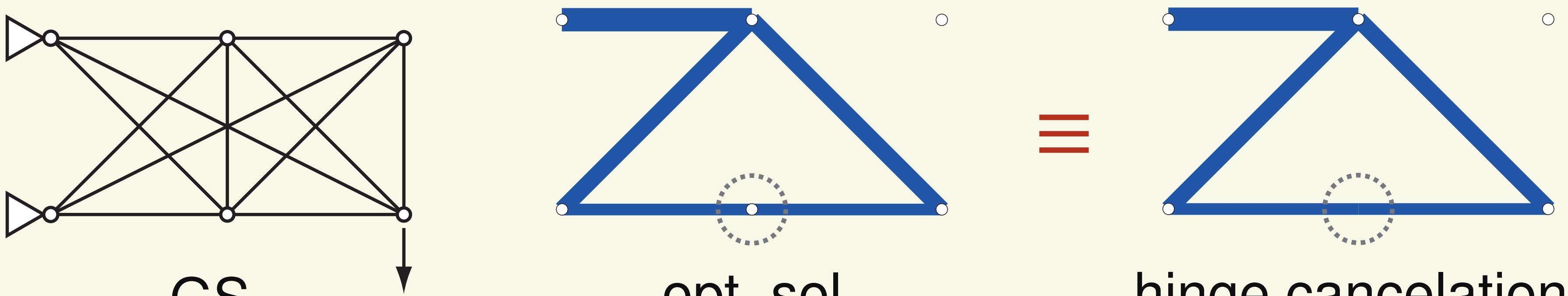
GS   opt. sol.   hinge cancelation

- chain (a set of sequential parallel bars)

  —can be replaced with a single longer bar

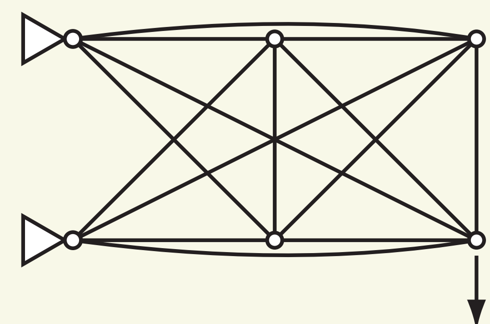
revisiting ground structure

- conventional compliance min.:
  - Overlapping bars in GS are omitted *a priori*.



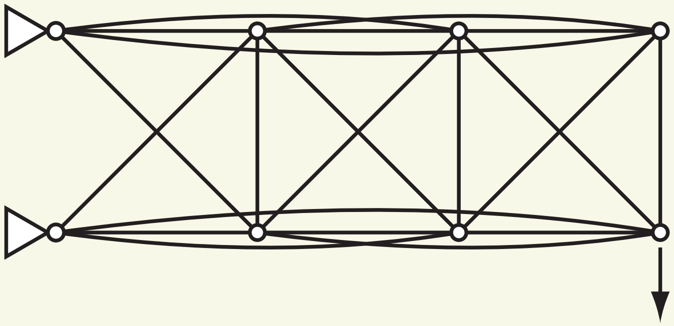
GS   opt. sol.   hinge cancelation

- chain (a set of sequential parallel bars)
  —can be replaced with a single longer bar
- #nodes:
  - "left" = 5   "right" = 4
  - "hinge cancelation" → a different solution
  - Overlapping bars in GS are not redundant.

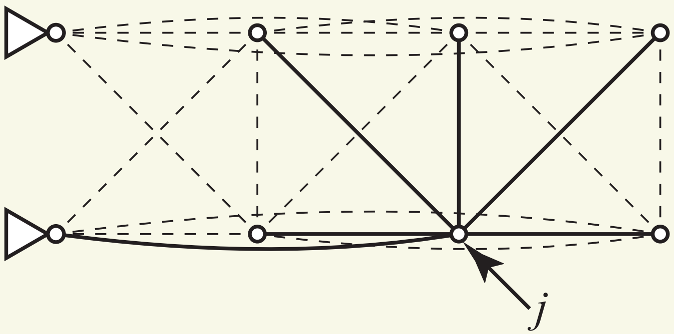


#node constraint

- GS w/ overlapping bars

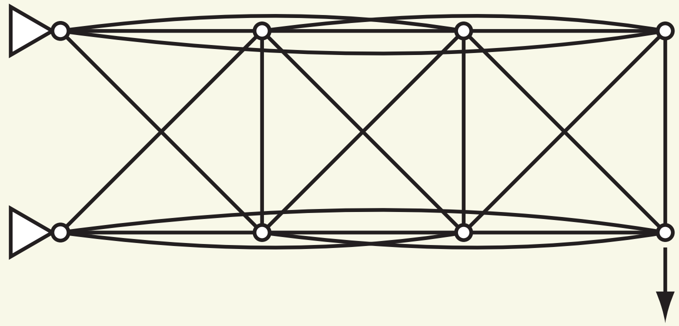


- z_j : sum of c-s areas of bars connecting to node j

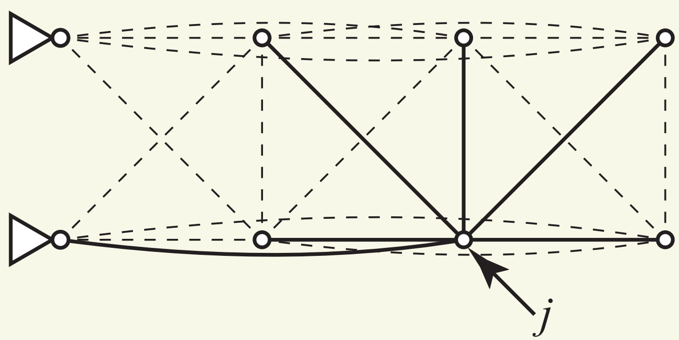


#node constraint

- GS w/ overlapping bars



- z_j : sum of c-s areas of bars connecting to node j

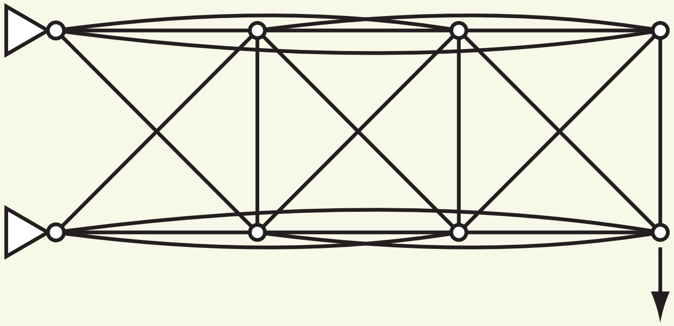


- n = upr. bnd., $\boldsymbol{z} = (z_1, \dots, z_l)^\top$

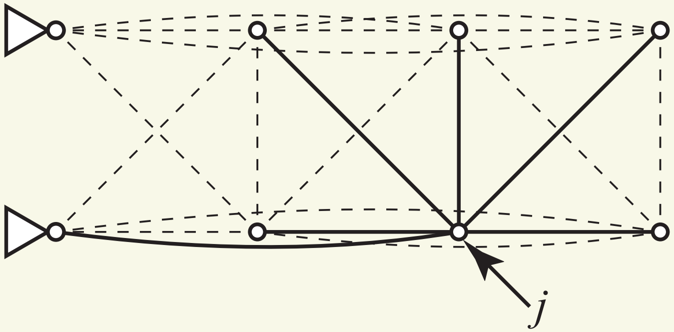
$$\|\boldsymbol{z}\|_0 \leq n \qquad (\ell_0\text{-norm} = \#\text{nonzero components})$$

#node constraint

- GS w/ overlapping bars



- z_j : sum of c-s areas of bars connecting to node j



- n = upr. bnd., $\boldsymbol{z} = (z_1, \dots, z_l)^\top$

$$\|\boldsymbol{z}\|_0 \le n \qquad (\ell_0\text{-norm} = \#\text{nonzero components})$$

- difficult (combinatorial) cstr. → a heuristic (based on ADMM)

ADMM = alternating direction method of multipliers

ADMM = alternating direction method of multipliers

- an algorithm for the convex optimization:

$$\begin{aligned} \text{Min.} \quad & f(\boldsymbol{x}) + g(\boldsymbol{z}) \\ \text{s. t.} \quad & A\boldsymbol{x} + B\boldsymbol{z} = \boldsymbol{c} \end{aligned}$$

  - f, g : convex $\qquad$ $\boldsymbol{x}, \boldsymbol{z}$: design variables

ADMM = alternating direction method of multipliers

- an algorithm for the convex optimization:

$$\begin{aligned} \text{Min.} \quad & f(\boldsymbol{x}) + g(\boldsymbol{z}) \\ \text{s. t.} \quad & A\boldsymbol{x} + B\boldsymbol{z} = \boldsymbol{c} \end{aligned}$$

  - f, g : convex $\boldsymbol{x}, \boldsymbol{z}$: design variables

- classical algorithm   [Glowinski & Marrocco '75], [Gabay & Mercier '76]
  - application in distributed optimization
  - a precursor: method of multipliers

ADMM = alternating direction method of multipliers

- an algorithm for the convex optimization:

$$\begin{array}{ll} \text{Min.} & f(\boldsymbol{x}) + g(\boldsymbol{z}) \\ \text{s. t.} & A\boldsymbol{x} + B\boldsymbol{z} = \boldsymbol{c} \end{array}$$

  - f, g : convex $\boldsymbol{x}, \boldsymbol{z}$: design variables
- classical algorithm   [Glowinski & Marrocco ’75], [Gabay & Mercier ’76]
  - application in distributed optimization
  - a precursor: method of multipliers
- recent attention
  - heuristic for nonconvex problems
    - sparse learning   [Chartrand ’07], [Chartrand & Wohlberg ’13]
    - learning on the Stiefel manifold   [Kanamori & Takeda ’14]
    - mixed-integer program   [Takapoui, Moehle, Boyd, & Bemporad ’17]

iteration of ADMM

- problem to be solved (f, g : convex):

$$\begin{aligned} \text{Min.} \quad & f(\boldsymbol{x}) + g(\boldsymbol{z}) \\ \text{s. t.} \quad & A\boldsymbol{x} + B\boldsymbol{z} = \boldsymbol{c} \end{aligned}$$

- augmented Lagrangian:

$$\begin{aligned} L_\rho(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{y}) = & f(\boldsymbol{x}) + g(\boldsymbol{z}) \\ & + \boldsymbol{y}^\top (A\boldsymbol{x} + B\boldsymbol{z} - \boldsymbol{c}) + \frac{\rho}{2} \| A\boldsymbol{x} + B\boldsymbol{z} - \boldsymbol{c} \|^2 \end{aligned}$$

  - $\boldsymbol{y}$: Lagrange multiplier $\quad$ $\rho > 0$: penalty parameter

- ADMM—alternating minimization w.r.t. primal variables:

$$\begin{aligned} \boldsymbol{x}^{k+1} & := \text{minimizer of } L_\rho(\boldsymbol{x}, \boldsymbol{z}^k; \boldsymbol{y}^k) \\ \boldsymbol{z}^{k+1} & := \text{minimizer of } L_\rho(\boldsymbol{x}^{k+1}, \boldsymbol{z}; \boldsymbol{y}^k) \\ \boldsymbol{y}^{k+1} & := \boldsymbol{y}^k + \rho(A\boldsymbol{x}^{k+1} + B\boldsymbol{z}^{k+1} - \boldsymbol{c}) \end{aligned}$$

formulation & augmented Lagrangian

- truss topology optimization w/ node constraint:

$$\begin{aligned} \text{Min.} \quad & \pi(\boldsymbol{x}) && \text{(compliance)} \\ \text{s. t.} \quad & \boldsymbol{x} \geq \boldsymbol{0}, \ \boldsymbol{c}^\top \boldsymbol{x} \leq V && \text{(nonnegative \& volume cstr.)} \\ & \boldsymbol{z} = S\boldsymbol{x} && \text{(def. of } \boldsymbol{z}) \\ & \|\boldsymbol{z}\|_0 \leq n && (\ell_0\text{-norm cstr.}) \end{aligned}$$

- augmented Lagrangian:

$$L_\rho(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{y}) = \pi(\boldsymbol{x}) + \boldsymbol{y}^\top (S\boldsymbol{x} - \boldsymbol{z}) + \frac{\rho}{2}\|S\boldsymbol{x} - \boldsymbol{z}\|^2$$
$$\text{s. t. (ineq. cstr.)}$$

  - $\boldsymbol{y}$: Lagrange multiplier $\qquad \rho > 0$: penalty parameter

- scaling of variable $\boldsymbol{v} := \boldsymbol{y}/\rho$ [Boyd, Parikh, Chu, Peleato, & Eckstein ’10]

$$L_\rho(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{v}) = \pi(\boldsymbol{x}) + \frac{\rho}{2}\|S\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{v}\|^2 - \frac{\rho}{2}\|\boldsymbol{v}\|^2$$

  - $\rightarrow$ scaled form of ADMM (shorter formula)

applying ADMM

- augmented Lagrangian (scaled):

$$L_\rho(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{v}) = \pi(\boldsymbol{x}) + \frac{\rho}{2}\|S\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{v}\|^2 - \frac{\rho}{2}\|\boldsymbol{v}\|^2$$

$$\text{s. t.} \quad \boldsymbol{x} \geq \boldsymbol{0}, \ \boldsymbol{c}^\top \boldsymbol{x} \leq V, \ \|\boldsymbol{z}\|_0 \leq n$$

- update of $\boldsymbol{z}$:

$$\text{Min.} \quad \frac{\rho}{2}\|(S\boldsymbol{x}^k + \boldsymbol{v}^k) - \boldsymbol{z}\|^2$$

$$\text{s. t.} \quad \|\boldsymbol{z}\|_0 \leq n$$

  - nonconvex, but easily computable
    - projection of point $S\boldsymbol{x}^k + \boldsymbol{v}^k$ onto set $\{\boldsymbol{z} \mid \|\boldsymbol{z}\|_0 \leq n\}$
      - keeping n largest magnitude components;
      - zeroing out the other components.

[Boyd, Parikh, Chu, Peleato, & Eckstein '10]

applying ADMM

- augmented Lagrangian (scaled):

$$L_\rho(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{v}) = \pi(\boldsymbol{x}) + \frac{\rho}{2}\|S\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{v}\|^2 - \frac{\rho}{2}\|\boldsymbol{v}\|^2$$
$$\text{s. t.} \quad \boldsymbol{x} \geq \boldsymbol{0}, \ \boldsymbol{c}^\top \boldsymbol{x} \leq V, \ \|\boldsymbol{z}\|_0 \leq n$$

- update of $\boldsymbol{x}$:

$$\text{Min.} \quad \pi(\boldsymbol{x}) + \frac{\rho}{2}\|S\boldsymbol{x} - \boldsymbol{z}^k + \boldsymbol{v}^k\|^2$$
$$\text{s. t.} \quad \boldsymbol{x} \geq \boldsymbol{0}, \ \boldsymbol{c}^\top \boldsymbol{x} \leq V$$

  - conventional compliance min. + “quadratic penalty”
    - can be solved easily via convex optimization.

applying ADMM

- augmented Lagrangian (scaled):

$$L_\rho(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{v}) = \pi(\boldsymbol{x}) + \frac{\rho}{2}\|S\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{v}\|^2 - \frac{\rho}{2}\|\boldsymbol{v}\|^2$$
$$\text{s. t.} \quad \boldsymbol{x} \geq \boldsymbol{0}, \ \boldsymbol{c}^\top \boldsymbol{x} \leq V, \ \|\boldsymbol{z}\|_0 \leq n$$

- update of $\boldsymbol{x}$:

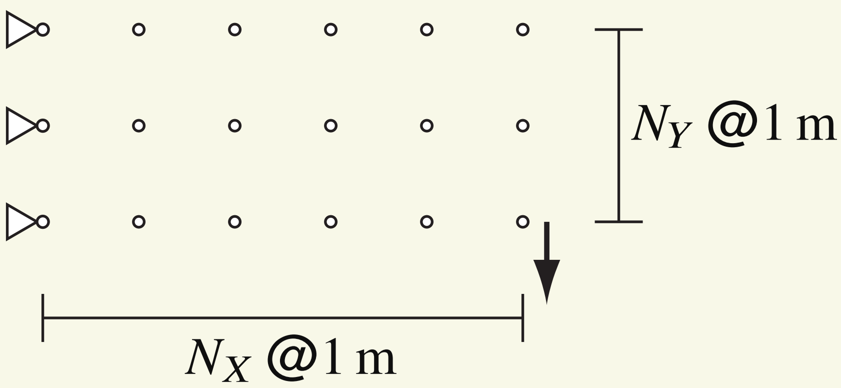
$$\text{Min.} \quad \pi(\boldsymbol{x}) + \frac{\rho}{2}\|S\boldsymbol{x} - \boldsymbol{z}^k + \boldsymbol{v}^k\|^2$$
$$\text{s. t.} \quad \boldsymbol{x} \geq \boldsymbol{0}, \ \boldsymbol{c}^\top \boldsymbol{x} \leq V$$

  - conventional compliance min. + “quadratic penalty”
    - can be solved easily via convex optimization.

- In ADMM, ℓ_0-norm cstr. is handled separately from the others.
  - Update of $\boldsymbol{z}$ (s. t. ℓ_0-norm cstr.) becomes very easy.
  - For fixed $\boldsymbol{z}$, problem becomes as easy as the conventional compliance min.

num. ex. (small scale)

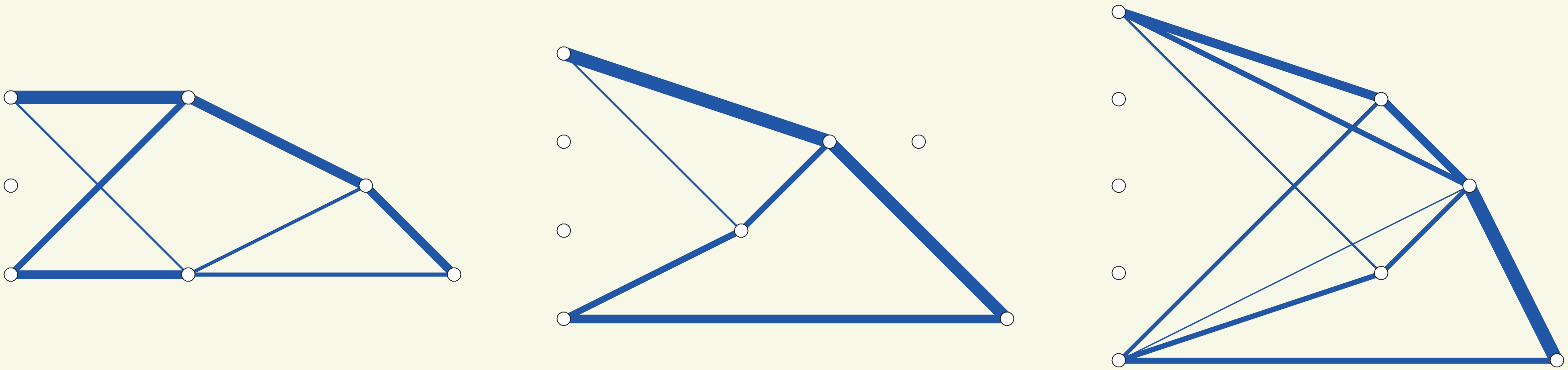
- problem setting



- $(N_X, N_Y) = (5, 2),\ (5, 3),\ (5, 4)$
  - #bars = #0-1 variables = 147, 264, 411

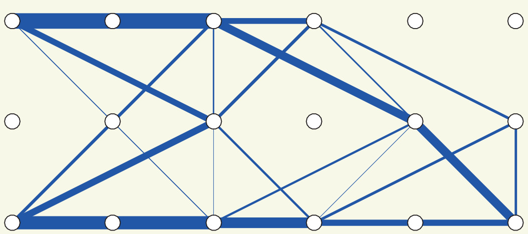
num. ex. (small scale)

- solutions obtained by the proposed ADMM (“#free nodes” ≤ 4)

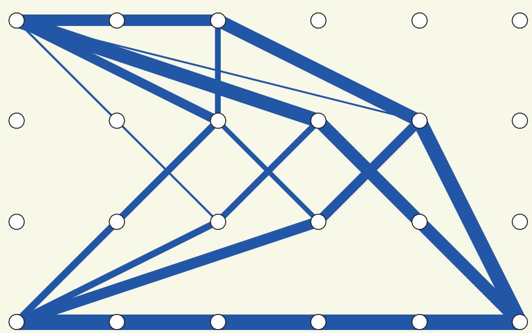


  - #iter ≤ 5

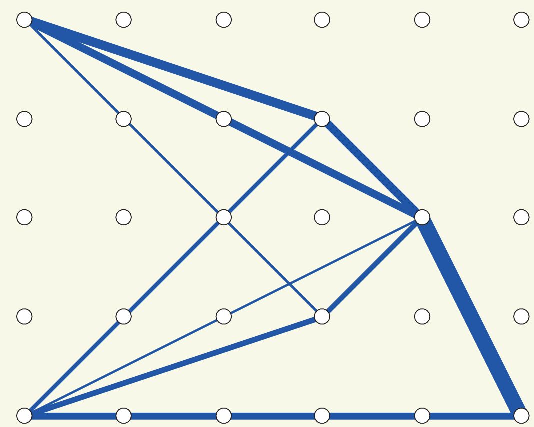
- opt. sol. w/o node constraint



“#free nodes” $= 9$



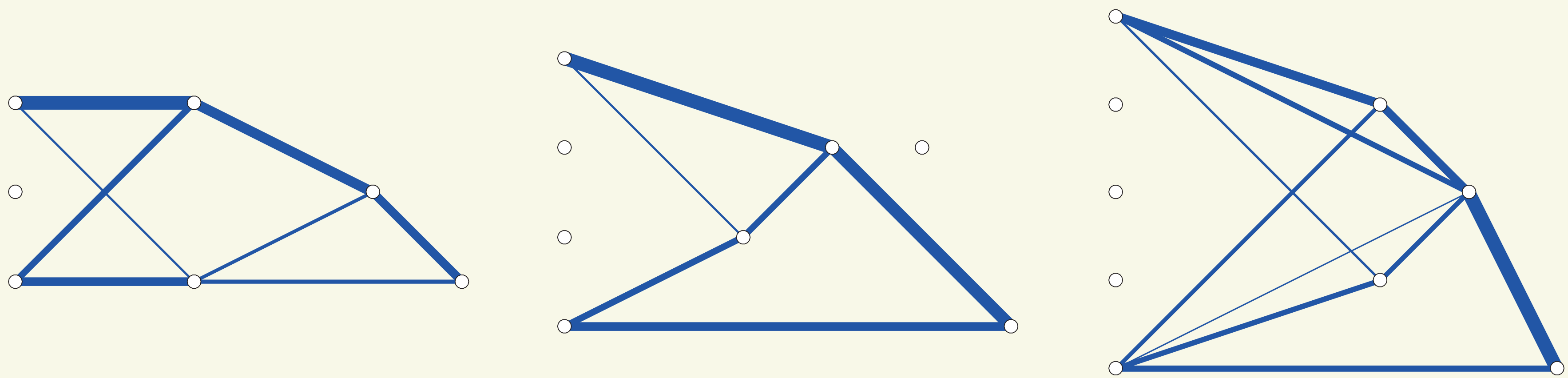
7



5

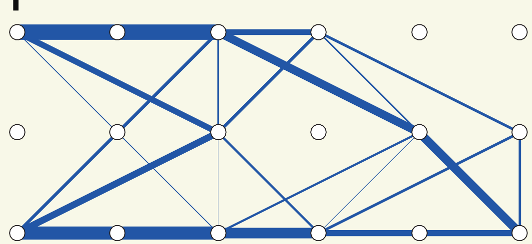
num. ex. (small scale)

- solutions obtained by the proposed ADMM (“#free nodes” ≤ 4)

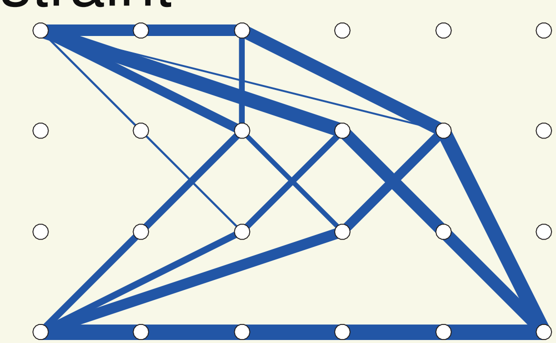


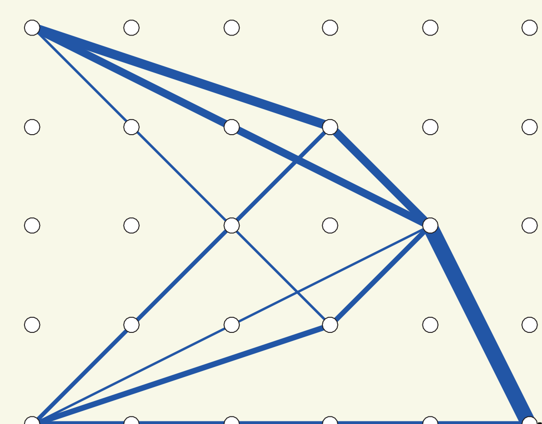
  - confirmed to be globally optimal
    (same obj as the solutions below)
    - multiple opt. sol.

- opt. sol. w/o node constraint



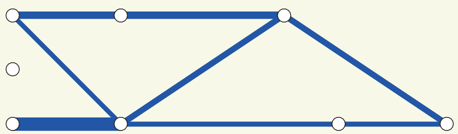
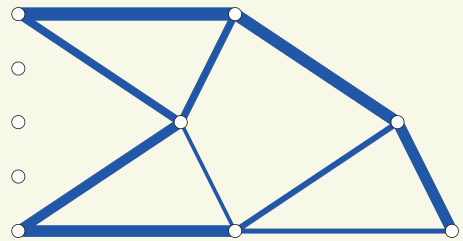
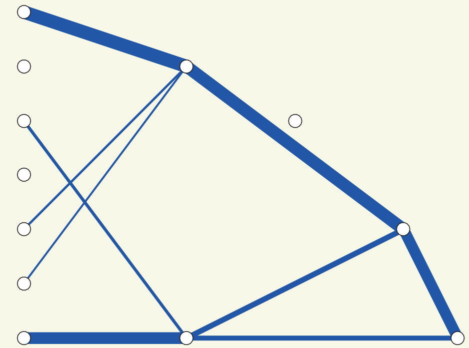
“#free nodes” $= 9$





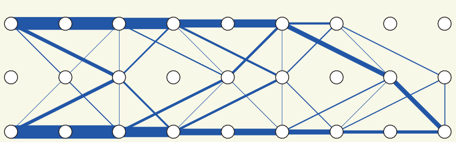
num. ex. (moderate scale, 1/2)

- solutions by the proposed ADMM ("#free nodes" ≤ 5)

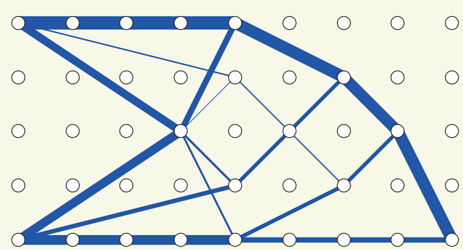




  - #bars in GS $= 273, 750, 1296$
  - #iter ≤ 10

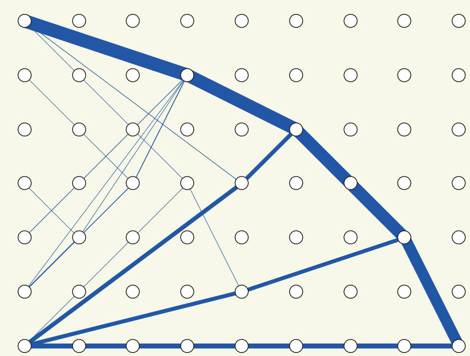
- opt. sol. w/o node constraint



"#free nodes" $= 15$



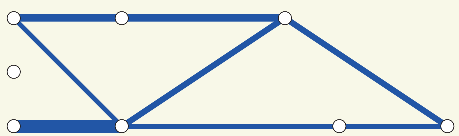
10



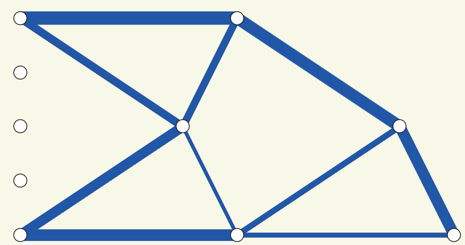
10

num. ex. (moderate scale, 1/2)

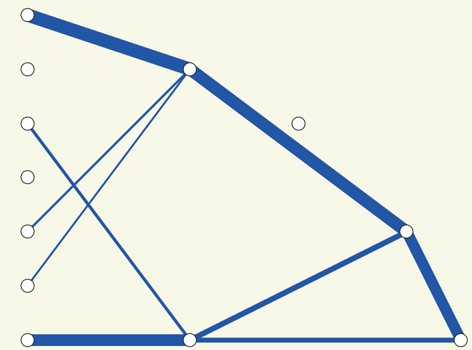
- solutions by the proposed ADMM w^* :



$w^*/\bar{w} = 1.072$

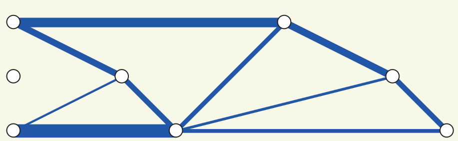


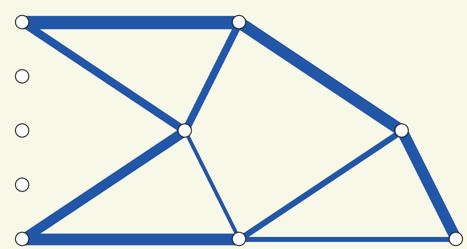
global opt.

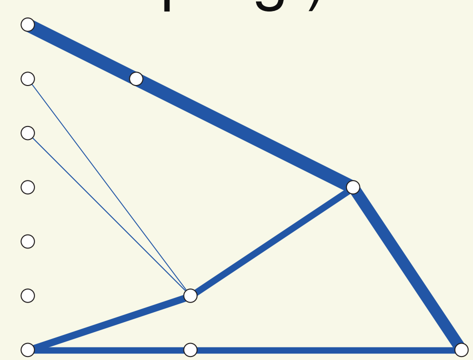


$w^*/\bar{w} = 1.038$

- global opt. sol. $\bar{w}$ (via mixed-integer 2nd-order cone prog.)

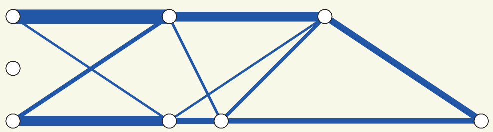
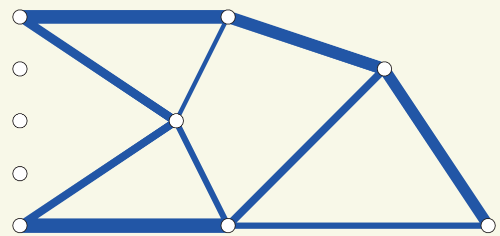
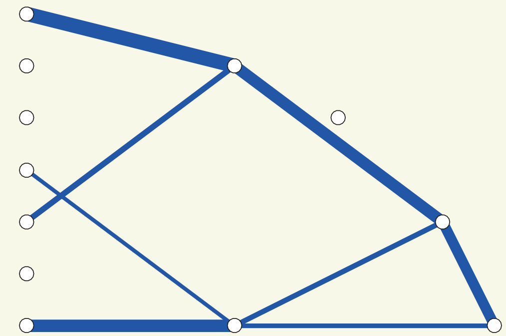






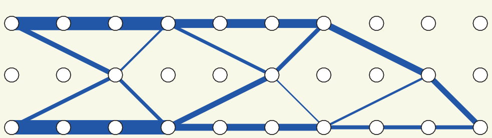
num. ex. (moderate scale, 2/2)

- solutions by the proposed ADMM (“#free nodes” ≤ 5)

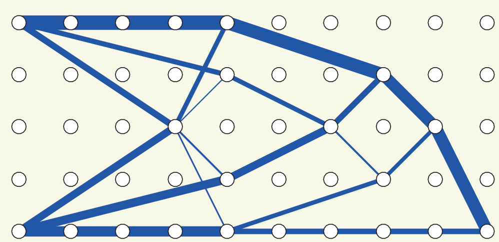




  - #bars in GS $= 315,\ 863,\ 1489$
  - #iter ≤ 10
  - no thin bars

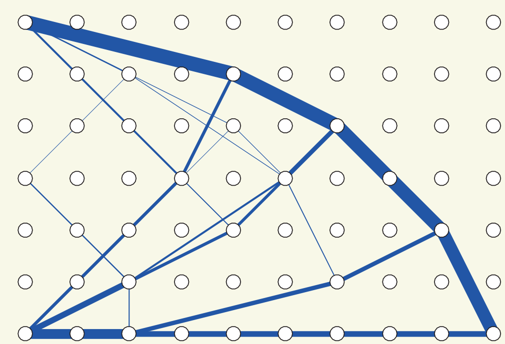
- opt. sol. w/o node constraint



“#free nodes” $= 8$



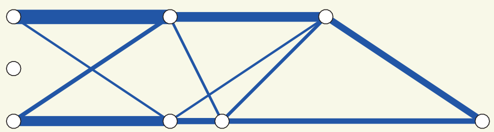
10



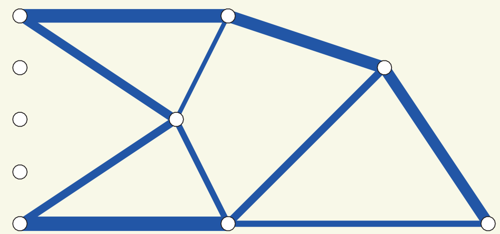
12

num. ex. (moderate scale, 2/2)

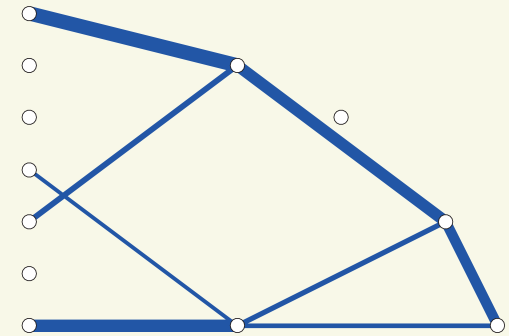
- solutions by the proposed ADMM w^* :



$w^*/\bar{w} = 1.070$

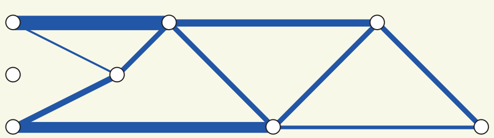


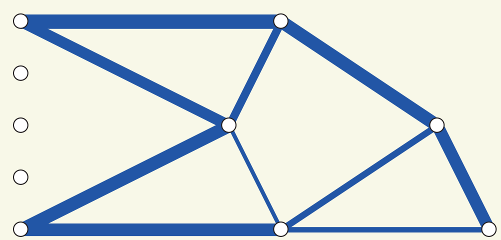
global opt.

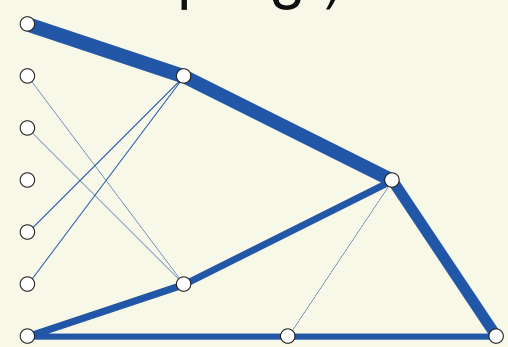


$w^*/\bar{w} = 1.060$

- global opt. sol. $\bar{w}$ (via mixed-integer 2nd-order cone prog.)







  - some thin bars

summary

- truss topology optimization w/ constraint on number of nodes
  - combinatorial feature
    - selection of nodes among candidates in a ground structure
    - ℓ_0-norm constraint
    - mixed-integer second-order cone programming
- ADMM (alternating direction method of multipliers)
  - heuristic for nonconvex optimization problems
    - simple algorithm
    - often shows fast convergence to a solution w/ high quality
    - depends on initial point & penalty parameter