On Multiplicity of Eigenvalues in Robust Compliance Optimization under Uncertain Loads

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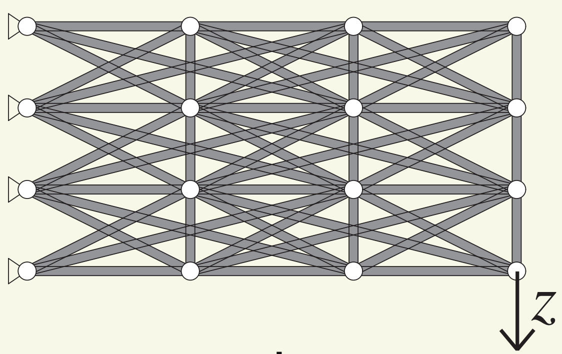
robust optimization

- nominal (i.e., conventional) compliance optimization

$$\begin{array}{ll} \min & compl(\boldsymbol{x}; \boldsymbol{z}) \\ \text{s. t.} & \boldsymbol{x} \geq \boldsymbol{0}, \quad vol(\boldsymbol{x}) \leq \bar{V} \end{array}$$

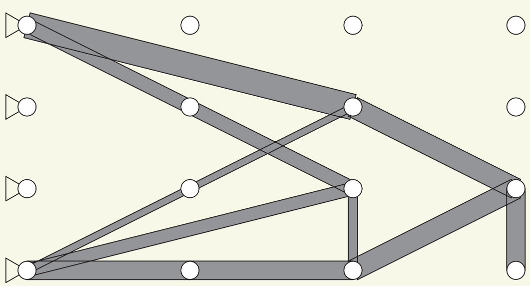
  - $\boldsymbol{x}$: cross-sectional areas $\qquad$ $\boldsymbol{z}$: data (e.g., external load)



ground struct.

$\longrightarrow$

optimize



nominal opt. sol.

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  - $\boldsymbol{x}$: cross-sectional areas z : data (e.g., external load)

- robust compliance optimization

$$\begin{aligned} \min \quad & \max\{compl(\boldsymbol{x}; z) \mid z \in \mathcal{U}\} \\ \text{s. t.} \quad & \boldsymbol{x} \geq \boldsymbol{0}, \quad vol(\boldsymbol{x}) \leq \bar{V} \end{aligned}$$

  - $\mathcal{U}$: uncertainty set   (e.g., set of uncertain loads)
  - obj. fcn.: worst value of compliance

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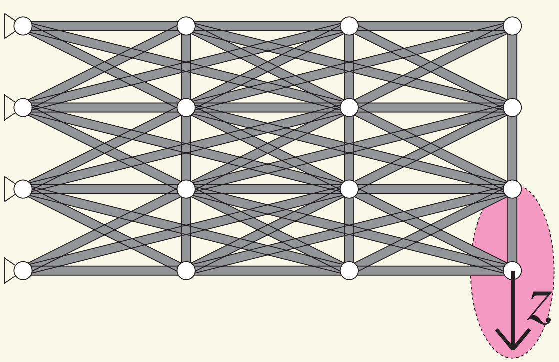
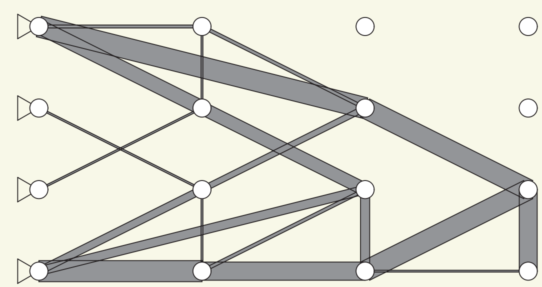


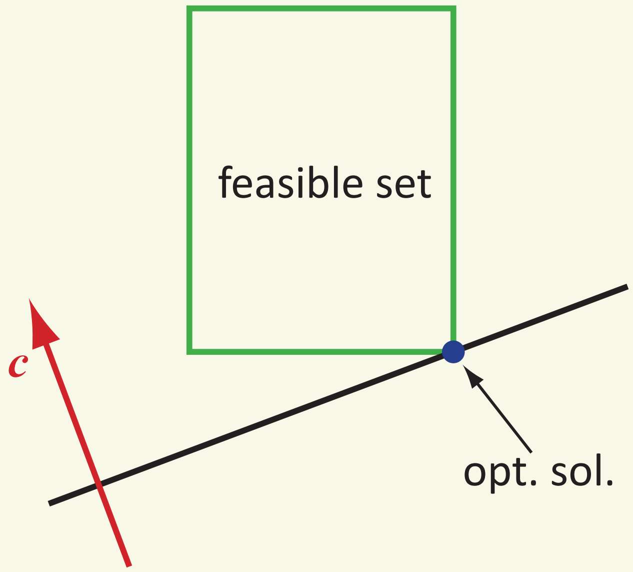
uncertainty: $z \in \mathcal{U}$

$\longrightarrow$ optimize



robust opt. sol.

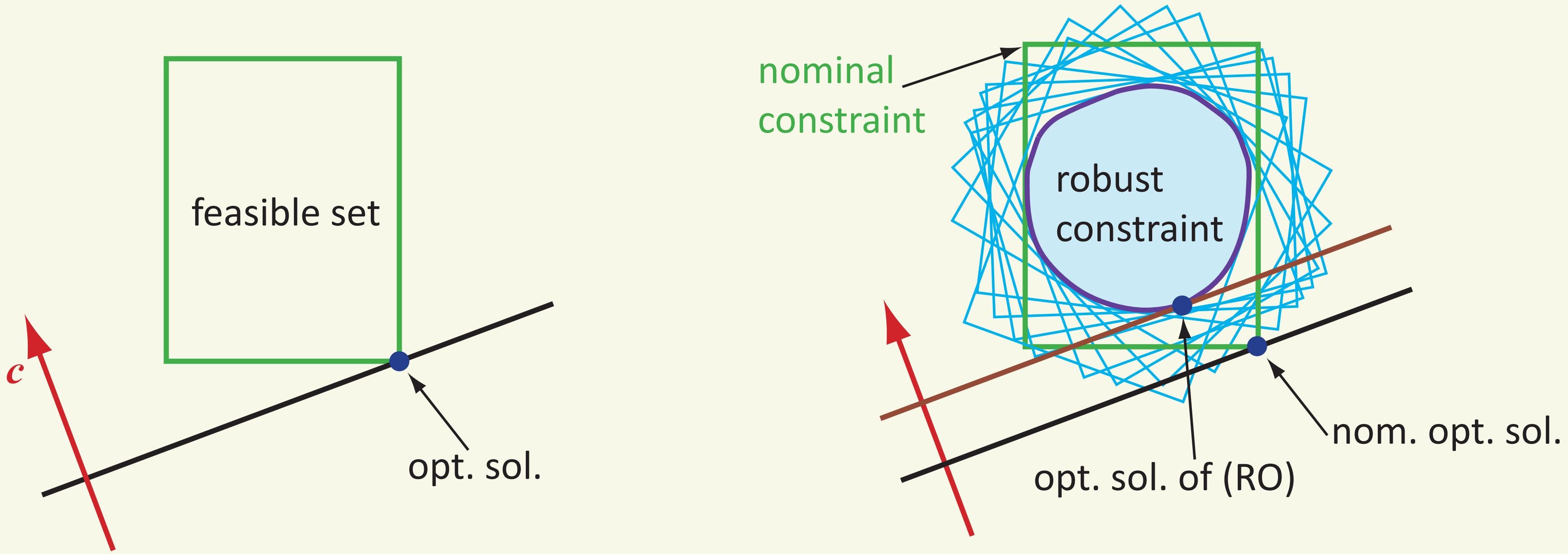
robust optimization



- nominal opt.:

$$\begin{array}{ll} \min & \boldsymbol{c}^{\top}\boldsymbol{x} \\ \mathrm{s.\,t.} & l_i \leq x_i \leq u_i \end{array}$$

robust optimization



- nominal opt.:

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\top} \boldsymbol{x} \\ \text{s.t.} \quad & l_i \le x_i \le u_i \end{aligned}$$

- robust opt.:

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\top} \boldsymbol{x} \\ \text{s.t.} \quad & l_i(\boldsymbol{z}) \le x_i \le u_i(\boldsymbol{z}) \quad (\forall \boldsymbol{z} \in \mathcal{U}) \end{aligned} \qquad \text{(RO)}$$

SDPs for robust optimization

- SDP = semidefinite programming — convex
  - compliance, load unc. [Ben-Tal & Nemirovski '97]
  - compliance, geometric unc. (safe approx.) [Hashimoto & K. '15]
- nonlinear SDP — nonconvex
  - stress cstr., load unc. [K. & Takewaki '06]
  - stiffness unc. (safe approx.) [Guo, Bai, Zhang, & Gao '09]
  - stiffness & length unc. (safe approx.) [Guo, Du, & Gao '11]

fundamental from linear algebra

- def.

$$B - C^{\top} A^{-1} C \;:\; \text{the Schur complement of } A \text{ in } \left[\begin{array}{c|c} A & C \\ \hline C^{\top} & B \end{array}\right]$$

  - A, B : symm. matrices

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- Lemma on the Schur complement
  - Assume A is p.d.
  - Then,

$$\left[\begin{array}{c|c} A & C \\ \hline C^\top & B \end{array}\right] \text{ is p.s.d.} \quad \Leftrightarrow \quad B - C^\top A^{-1} C \text{ is p.s.d.}$$

- p.d. = positive definite $\Leftrightarrow$ all eigenvalues > 0.
- p.s.d. = positive semidefinite $\Leftrightarrow$ all eigenvalues ≥ 0.

problem setting

- $\boldsymbol{f}$: external load
- uncertainty set

$$\mathcal{U} = \{Q\boldsymbol{e} \mid \|\boldsymbol{e}\| \leq 1\}$$

  - Q : constant matrix
    - info. of "nom. load & unc. load"

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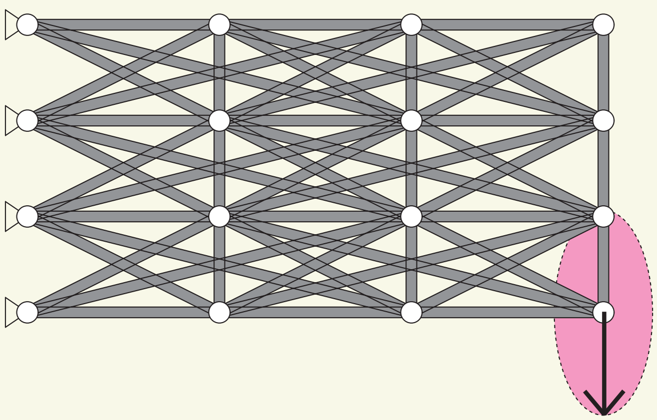

$$Q = \begin{bmatrix} \tilde{f} & 0 \\ 0 & \epsilon \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

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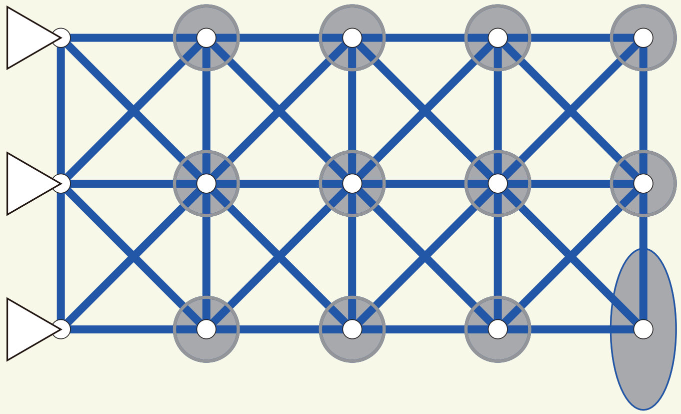

$$Q = \begin{bmatrix} \tilde{f} & 0 & 0 & \cdots & 0 \\ 0 & \epsilon & 0 & \cdots & 0 \\ 0 & 0 & \epsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \epsilon \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_d \end{bmatrix}$$

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- RO:

$$\begin{array}{ll} \min_{\boldsymbol{x}} & \max_{\boldsymbol{f}} \{ compl(\boldsymbol{x}; \boldsymbol{f}) \mid \boldsymbol{f} \in \mathcal{U} \} \\ \mathrm{s.\,t.} & \boldsymbol{x} \ge \boldsymbol{0}, \quad vol(\boldsymbol{x}) \le \bar{V} \end{array}$$

  - $\boldsymbol{x}$: c.-s. areas — design variables
  - Tractable reformulations are known.

tractable forms of (RO)

- three reformulations in literature
  - SDP formulation

    [Ben-Tal & Nemirovski '97]
  - min. of maximum eigenvalue (standard eig. prob.)

    [Takezawa, Nii, Kitamura, & Kogiso '11]
  - min. of maximum eigenvalue (generalized eig. prob.)

    [Cherkaev & Cherkaev '03, '08]
- equivalence $\leftarrow$ the Schur complement lemma

SDP to standard eig. prob.

- SDP formulation [Ben-Tal & Nemirovski '97]

$$w \geq (\text{worst compl.})$$
$$\Leftrightarrow \quad \left[\begin{array}{c|c} K(\boldsymbol{x}) & Q \\ \hline Q^{\top} & wI \end{array}\right] \text{ is p.s.d.} \qquad (\spadesuit)$$

- $K(\boldsymbol{x})$: stiffness matrix
- w : upr. bd. for worst compl.

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- Apply the Schur complement lemma (assuming $K(\boldsymbol{x})$ is invertible):

$$(\spadesuit) \quad \Leftrightarrow \quad wI - Q^{\top} K(\boldsymbol{x})^{-1} Q \text{ is p.s.d.}$$

$$\left[\begin{array}{c|c} A & C \\ \hline C^{\top} & B \end{array}\right] \text{ is p.s.d.} \quad \Leftrightarrow \quad B - C^{\top} A^{-1} C \text{ is p.s.d.}$$

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- $\Leftrightarrow \quad w \geq \lambda_{\max}(Q^{\top} K(\boldsymbol{x})^{-1} Q)$

- Therefore,

$$\text{(worst compl.)} = \lambda_{\max}(Q^{\top} K(\boldsymbol{x})^{-1} Q)$$

[Takezawa, Nii, Kitamura, & Kogiso '11]

SDP to generalized eig. prob.

- SDP formulation [Ben-Tal & Nemirovski '97]

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- Apply the Schur complement lemma (assuming $w > 0$):

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$$(\spadesuit) \quad \Leftrightarrow \quad K(\boldsymbol{x}) - Q^\top (wI)^{-1} Q \text{ is p.s.d.}$$

  - $\Leftrightarrow wK(\boldsymbol{x}) - QQ^\top$ is p.s.d. i.e., $\boldsymbol{z}^\top (wK(\boldsymbol{x}) - QQ^\top)\boldsymbol{z} \geq 0 \ (\forall \boldsymbol{z})$
  - $\Leftrightarrow w \geq \dfrac{\boldsymbol{z}^\top QQ^\top \boldsymbol{z}}{\boldsymbol{z}^\top K(\boldsymbol{x})\boldsymbol{z}} \ (\forall \boldsymbol{z})$ (Rayleigh–Ritz ratio)

- Therefore,

$$\text{(worst compl.)} = \text{max. eigenvalue of } (QQ^\top)\boldsymbol{z} = \lambda K(\boldsymbol{x})\boldsymbol{z}$$

[Cherkaev & Cherkaev '03, '08]

upshot

- SDP formulation [Ben-Tal & Nemirovski '97]

$$w \geq \text{(worst compl.)} \quad \Leftrightarrow \quad \left[\begin{array}{c|c} K(\boldsymbol{x}) & Q \\ \hline Q^{\top} & wI \end{array}\right] \text{ is p.s.d.} \qquad (\spadesuit)$$

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- two ways of application of the Schur complement lemma
  - (1): → standard eig. prob. [Takezawa, Nii, Kitamura, & Kogiso '11]

$$(\spadesuit) \quad \Leftrightarrow \quad wI - Q^{\top} K(\boldsymbol{x})^{-1} Q \text{ is p.s.d.}$$

  - (2): → generalized eig. prob. [Cherkaev & Cherkaev '03, '08]

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on multiple eigenvalue

- the three formulations — related to eigenvalues
- eigenvalue optimization
  - In general, an opt. sol. often has multiple eigenvalues.
    - e.g., frequency, buckling load,...
  - ...How about (RO)?

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- the three formulations — related to eigenvalues
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  - In general, an opt. sol. often has multiple eigenvalues.
    - e.g., frequency, buckling load,...
  - ...How about (RO)? $\rightarrow$ Yes!
    - examples w/ simple eigenvalues

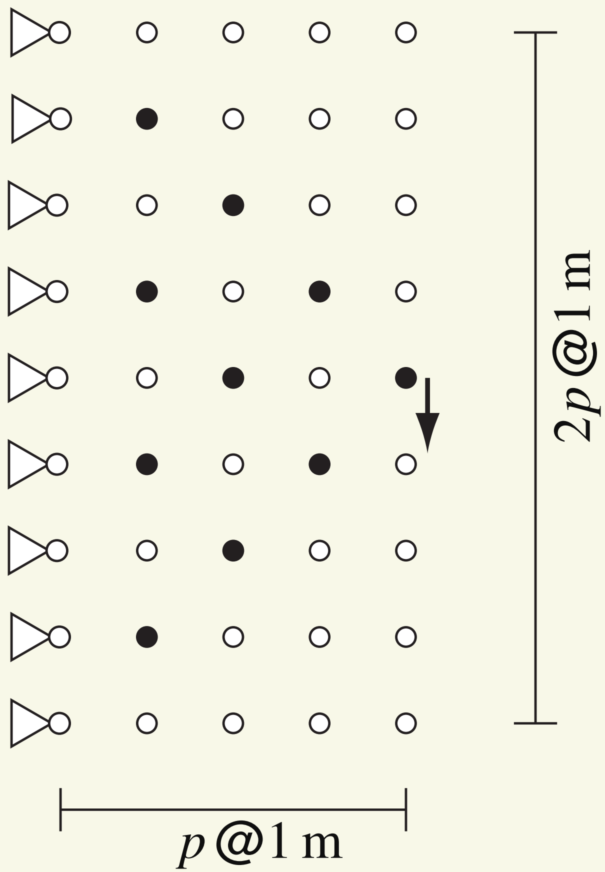
      [Takezawa, Nii, Kitamura, & Kogiso '11]

      [Brittain, Silva, & Tortorelli '12]
    - an example w/ fivefold eigenvalue

      [Herskovits, Freire, Tanaka Fo, & Canelas '11]
    - ...but more.

a series of problem instances

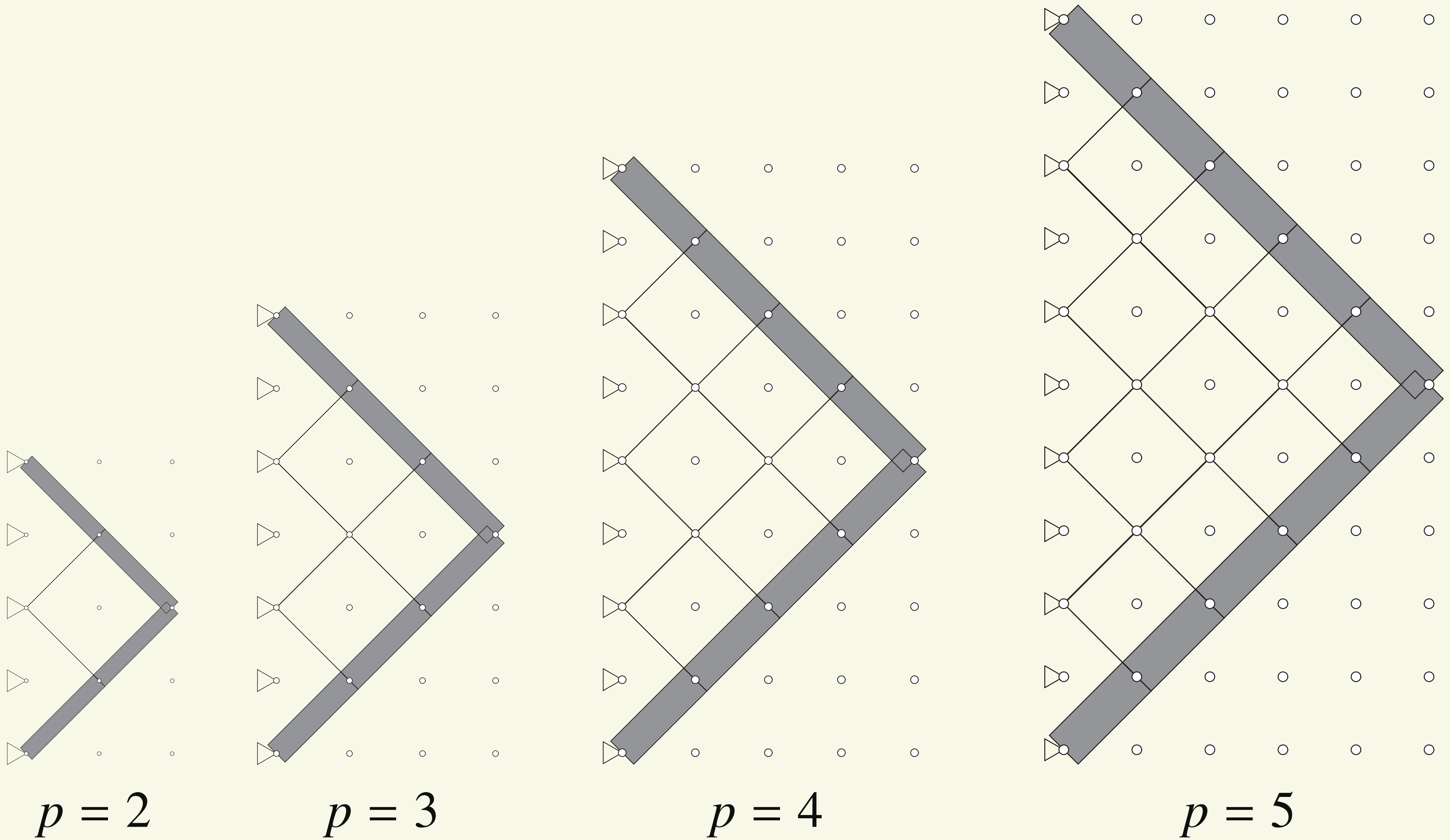
- truss (ground structure)



- Any two nodes are connected by a member.
  - Overlapping members have been removed.
- The problem size → large, when p → large.

a series of problem instances

- robust optimal solutions:



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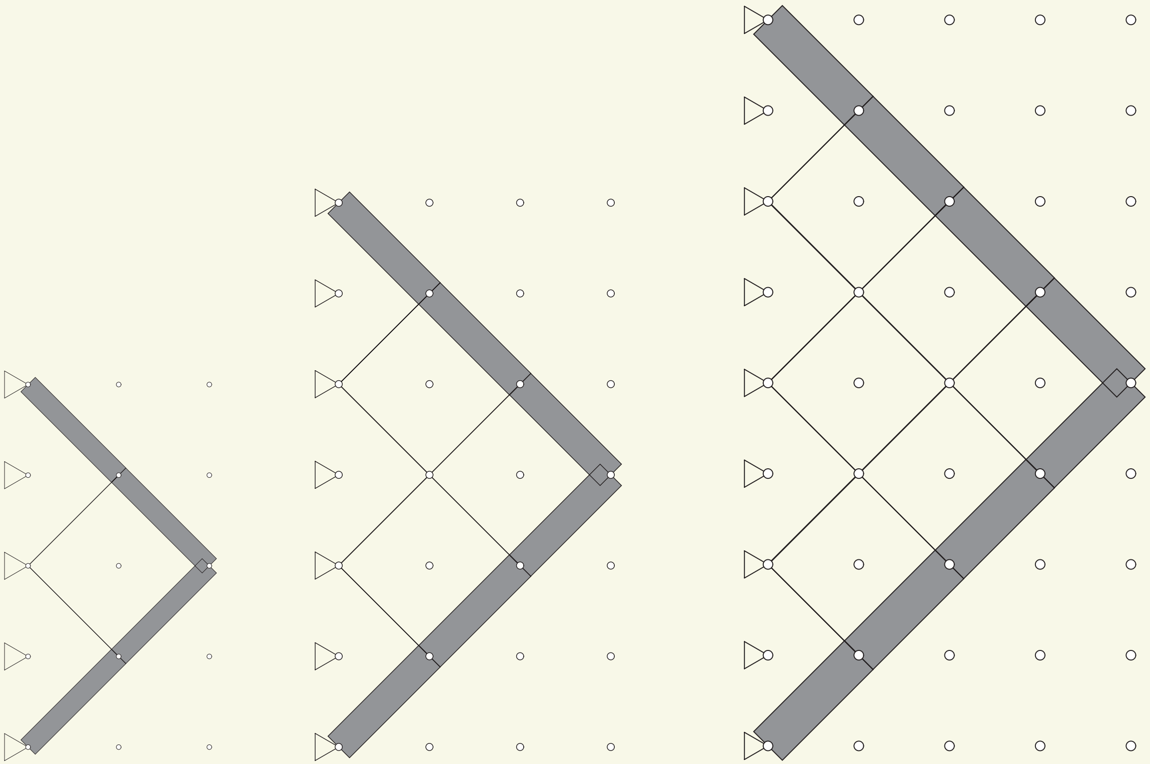


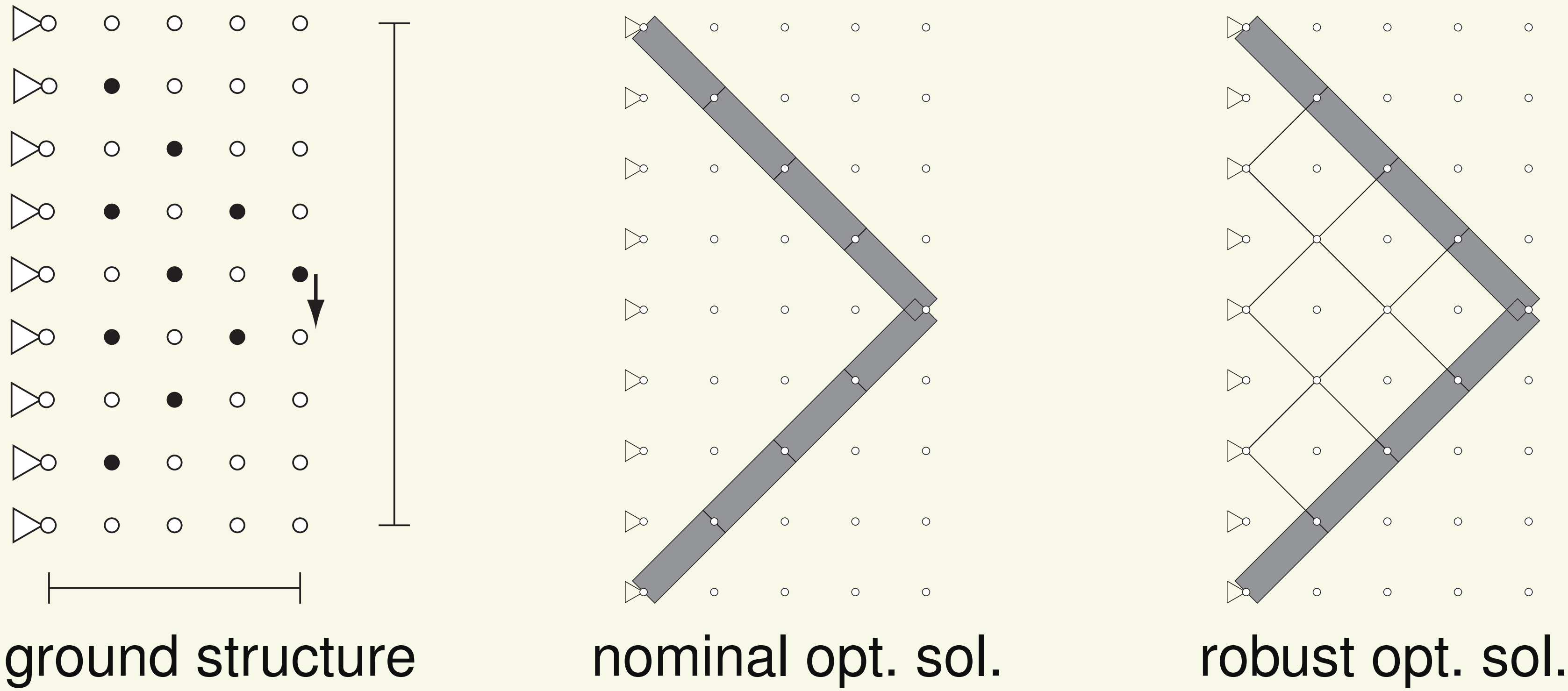
$p = 2$ $p = 3$ $p = 4$

- multiplicity of eigenvalues $= 2p - 1$
  - As p increases, multiplicity increases.

Arbitrary large multiplicity is possible.

source of multiplicity

- $p = 4$



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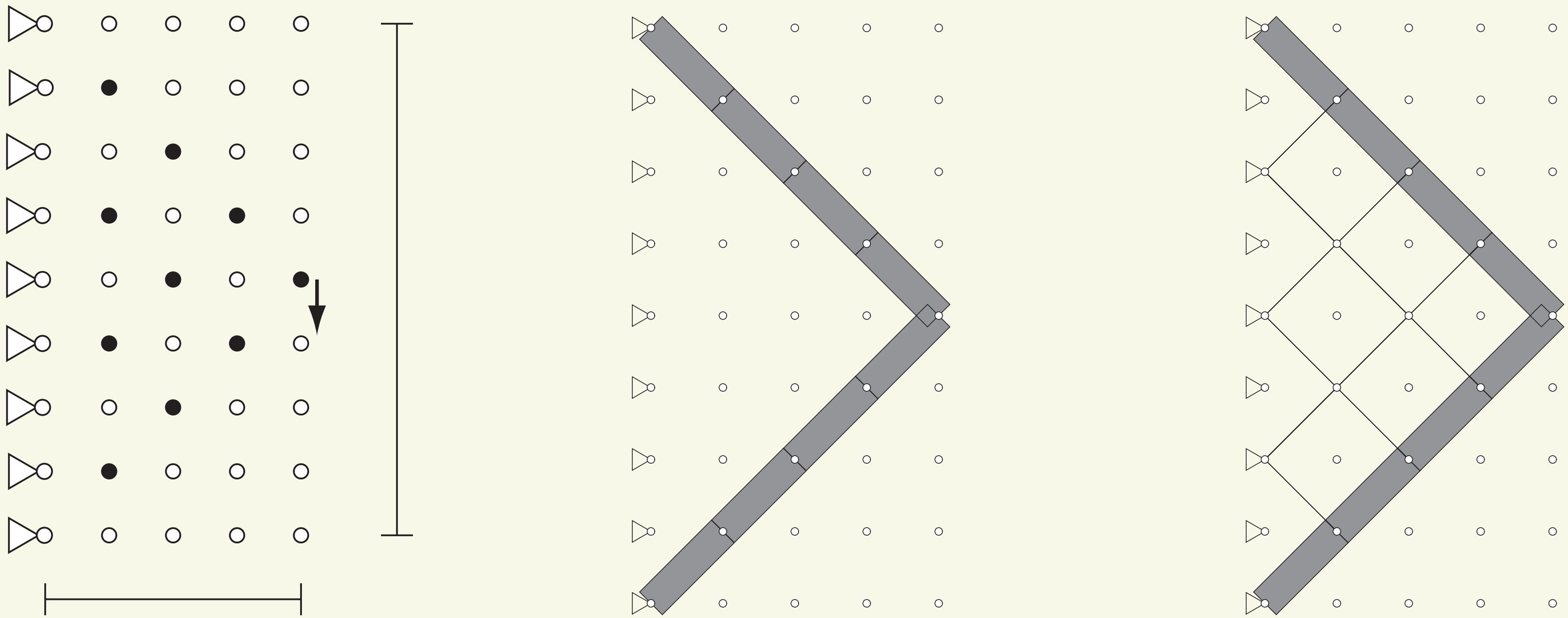

ground structure　　nominal opt. sol.　　robust opt. sol.

- multiple eigenvalues $\leftrightarrows$ worst-case loadings

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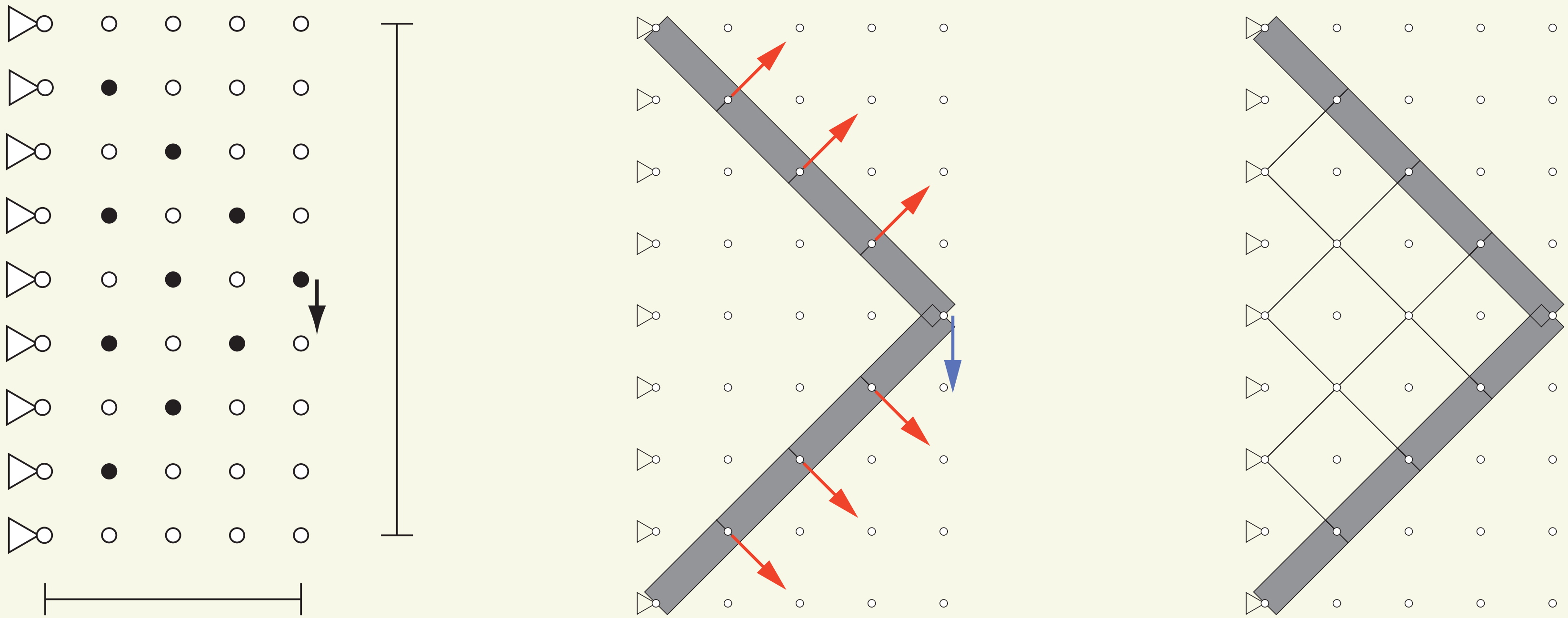

ground structure　　nominal opt. sol.　　robust opt. sol.

- multiple eigenvalues ⇆ worst-case loadings
  - 1 nominal load + $2(p-1)$ uncertain loads @ unstable nodes
    $= 2p - 1$ multiplicity

conclusions

- robust compliance optimization
  - ellipsoidal uncertainty in external load
  - worst-case compliance $\quad \rightarrow$ Min.
- three existing formulations
  - semidefinite programming [Ben-Tal & Nemirovski '97]
  - min. of max. eig. (standard eig. prob.) [Takezawa *et al.* '11]
  - min. of max. eig. (generalized eig. prob.) [Cherkaev & Cherkaev '03, '08]
  - $\rightarrow$ equivalence: explained via the Schur complement lemma
- multiplicity of eigenvalues at an opt. sol.
  - $\rightarrow$ a series of problems with arbitrary large multiplicity
  - multiple eigenvalues $\leftrightarrows$ multiple worst-case loadings
  - non-differentiability