2 次錐相補性問題に基づく
摩擦を伴う接触問題の増分解析法

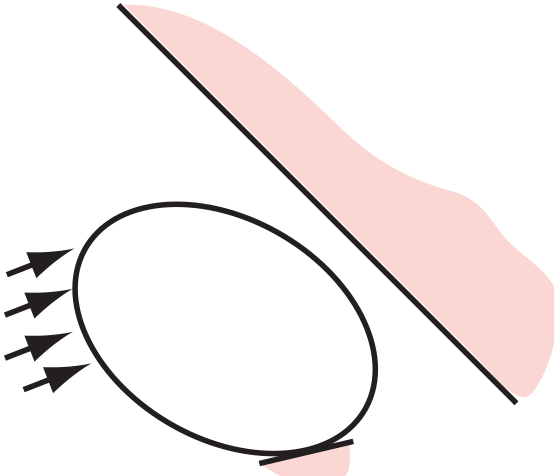
寒野 善博 [†]　João A.C. Martins[‡]　António Pinto da Costa[‡]

[†] 京都大学大学院（都市環境工学専攻）

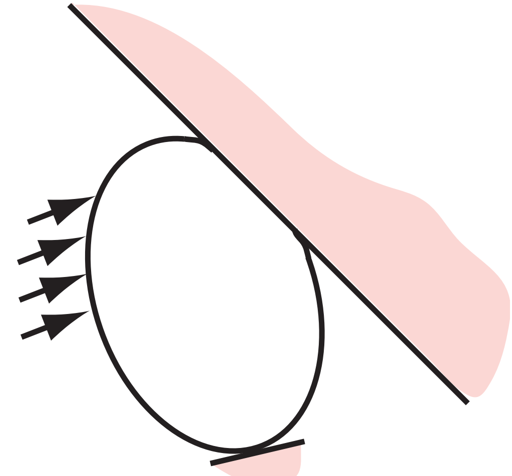
[‡] Instituto Superior Técnico (Departamento de Engenharia Civil and ICIST)

接触問題(contact problem)

- 片側接触, 弾性体と剛体の接触



初期状態



釣合状態

- 接触するか, 否か
- 接触するならば, どの範囲で接触するか?
  - 境界条件の一部 (変位拘束の有無, 反力) が未知

接触問題

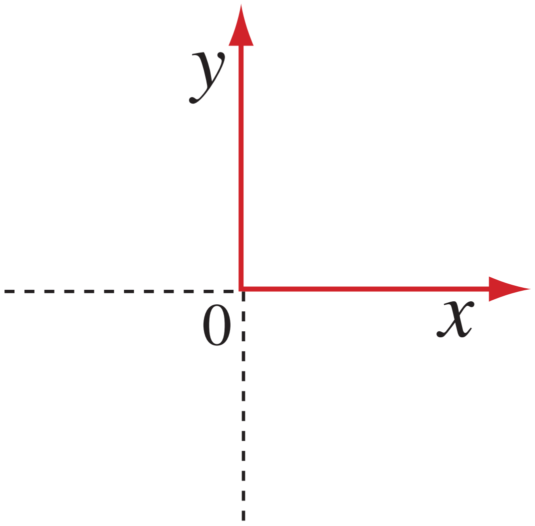
- **接触問題の分類**
  - **(摩擦のない) 接触問題**
    (frictionless) contact problem
  - **摩擦問題**
    friction problem
  - **摩擦のある接触問題**
    frictional contact problem
- **摩擦則**
  - Coulomb **摩擦則**
  - regularization
  - superelliptical friction rule [Hjiaj–Feng–de Saxcé–Mróz 04]

3 次元摩擦・接触問題の解法

- 拡張 Lagrangian 法
  - [Alart & Curinier 91] [Simo & Laersen 92] [de Saxcé & Feng 91]
  - ペナルティ法
- bi-potential 法 [de Saxcé & Feng 98]
- Gauss–Seidel 法 [Jourdan, Alart & Jean 98]
- nonsmooth Newton 法 [Johansen & Klarbring 00]
- trial-and-error
- 相補性問題
  - 線形相補性問題 +Lemke 法 [Klarbring and Bjorkman 92]
  - 2 次錐相補性問題 + 平滑化法

相補性条件(Complementarity condition)

$$\boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{y} \geq \boldsymbol{0}, \quad \boldsymbol{x}^\top \boldsymbol{y} = 0$$



相補性問題 (Complementarity problem)

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}),$$
$$\boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{y} \geq \boldsymbol{0}, \quad \boldsymbol{x}^\top \boldsymbol{y} = 0$$

- f: 線形 $\Longrightarrow$ **線形相補性問題 (LCP)**
- f: 非線形 $\Longrightarrow$ **非線形相補性問題 (NLCP)**

相補性問題 (Complementarity problem)

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- f: 線形 $\Longrightarrow$ 線形相補性問題 (LCP)
- f: 非線形 $\Longrightarrow$ 非線形相補性問題 (NLCP)

2 次錐相補性問題:

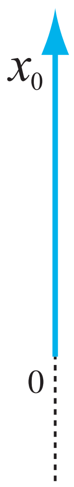
$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}),$$
$$\boldsymbol{x} \in \mathbf{L}^n_+, \quad \boldsymbol{y} \in \mathbf{L}^n_+, \quad \boldsymbol{x}^\top \boldsymbol{y} = 0$$

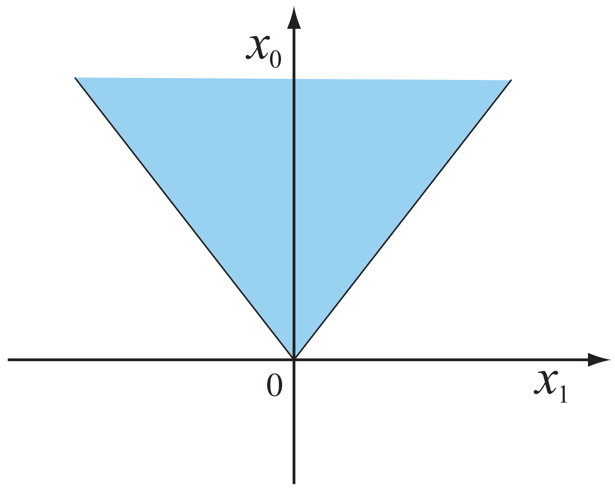
2次錐(second-order cone)

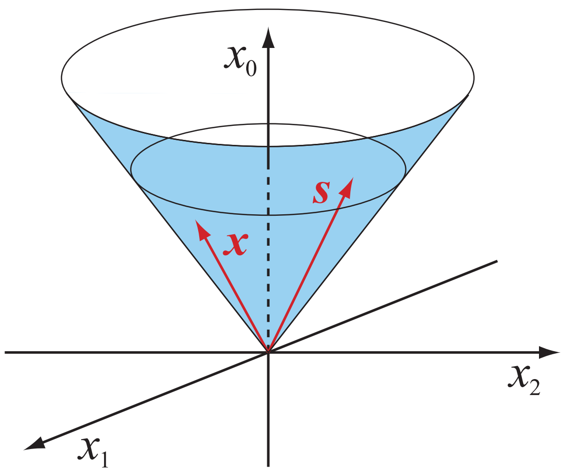
$$\boldsymbol{x} = (x_0, \boldsymbol{x}_1) \in \mathbf{R}^n$$

$$\mathbf{L}^n_+ = \{(x_0, \boldsymbol{x}_1) |\, x_0 \geq \|\boldsymbol{x}_1\|\}$$: **2次錐**

$$\|\boldsymbol{x}_1\| = (\boldsymbol{x}_1^\top \boldsymbol{x}_1)^{1/2}$$: Euclid **ノルム**







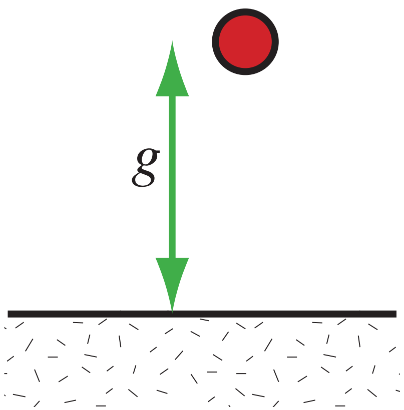
$\mathbf{L}^1_+$ $\mathbf{L}^2_+$: **2次元の場合** $\mathbf{L}^3_+$: **3次元の場合**

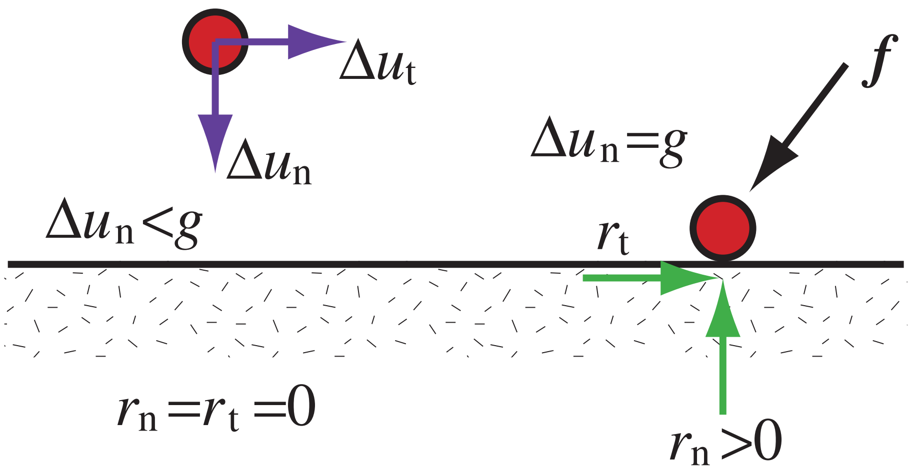
片側接触

非貫通条件 (non-penetration condition):

$$g - \Delta u_{\mathrm{n}} > 0 \quad \Longrightarrow \quad r_{\mathrm{n}} = 0 \qquad \text{: free}$$

$$r_{\mathrm{n}} > 0 \quad \Longrightarrow \quad g - \Delta u_{\mathrm{n}} = 0 \qquad \text{: contact}$$





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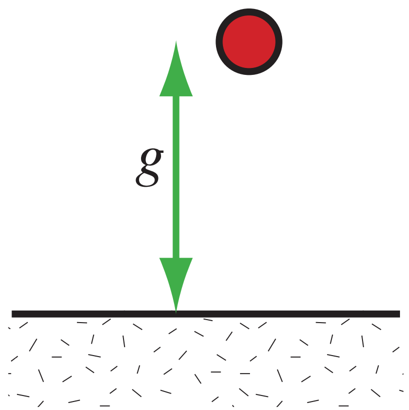
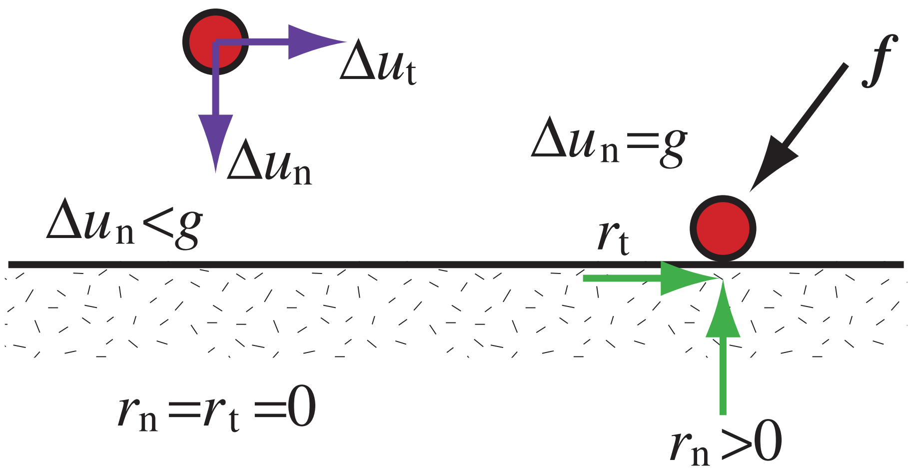




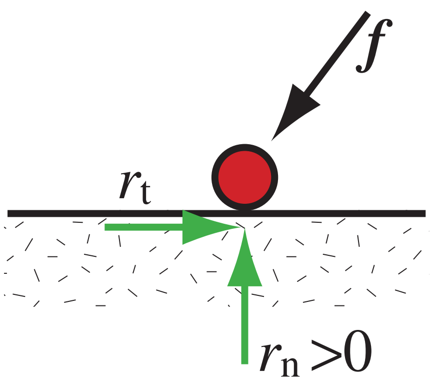
相補性条件:

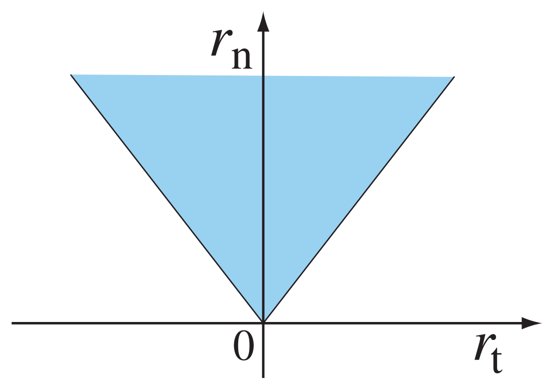
$$g - \Delta u_{\mathrm{n}} \geq 0, \quad r_{\mathrm{n}} \geq 0, \quad (g - \Delta u_{\mathrm{n}}) r_{\mathrm{n}} = 0$$

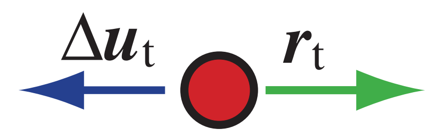
Coulomb 摩擦則

$$\mu r_{\mathrm{n}} \geq |r_{\mathrm{t}}|$$

$$\Delta \boldsymbol{u}_{\mathrm{t}} = -\alpha \boldsymbol{r}_{\mathrm{t}}, \quad \begin{cases} \alpha > 0 & \text{(slip)} \\ \alpha = 0 & \text{(stick)} \end{cases}$$





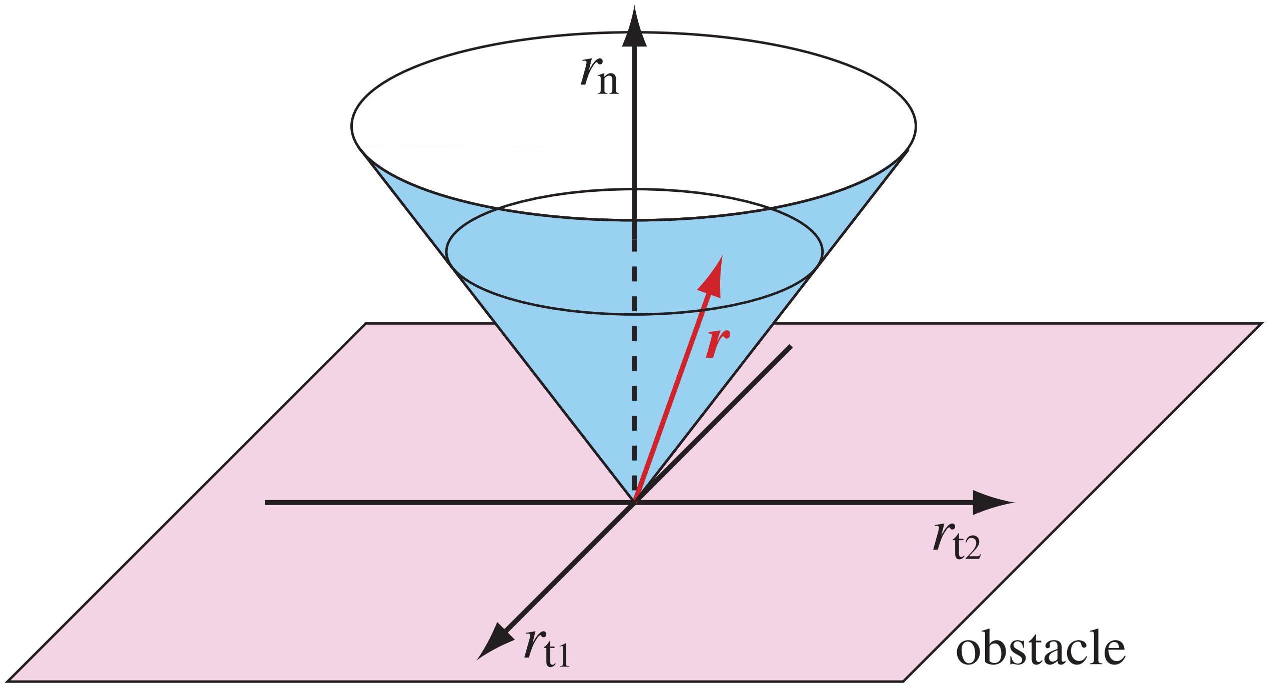


μ : **摩擦係数**

$(r_{\mathrm{n}}, r_{\mathrm{t}})$: **反力** (normal, tangential)

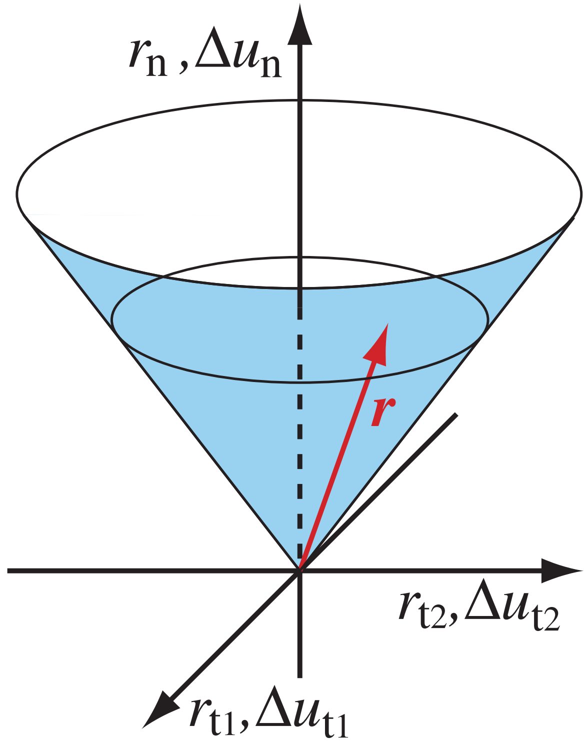
Friction cone

- Coulomb 摩擦則: $\mu r_{\mathrm{n}} \geq \|\boldsymbol{r}_{\mathrm{t}}\|$

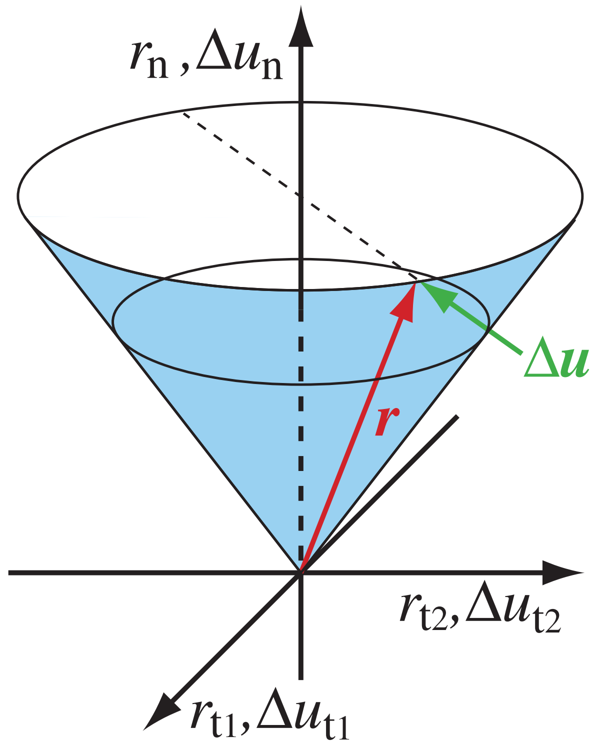


Friction cone

- Coulomb 摩擦則: $\mu r_{\mathrm{n}} \geq \|\boldsymbol{r}_{\mathrm{t}}\|$



stick



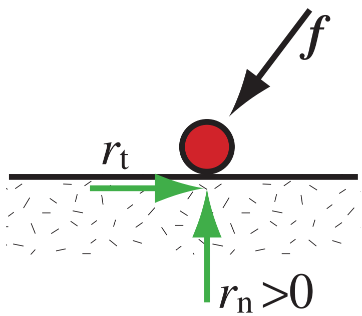
slip

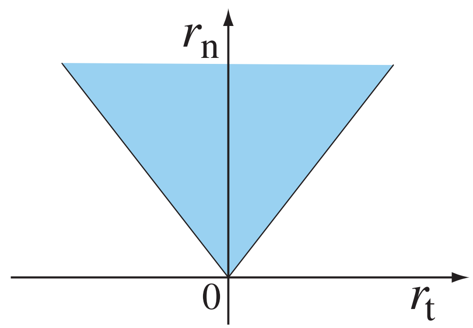
- non-associated law

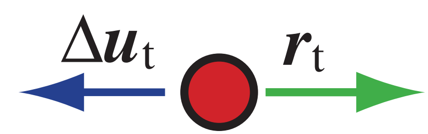
Coulomb 摩擦則

$$\mu r_{\rm n} \geq \|\boldsymbol{r}_{\rm t}\|$$

$$\Delta \boldsymbol{u}_{\rm t} = -\alpha \boldsymbol{r}_{\rm t}, \quad \begin{cases} \alpha > 0 & \text{(slip)} \\ \alpha = 0 & \text{(stick)} \end{cases}$$







[Duvaut & Lions 76]:

$$\mu r_{\rm n} \geq \|\boldsymbol{r}_{\rm t}\|,$$

$$\begin{pmatrix} \mu r_{\rm n} \\ \boldsymbol{r}_{\rm t} \end{pmatrix}^{\top} \begin{pmatrix} \|\Delta \boldsymbol{u}_{\rm t}\| \\ \Delta \boldsymbol{u}_{\rm t} \end{pmatrix} = 0$$

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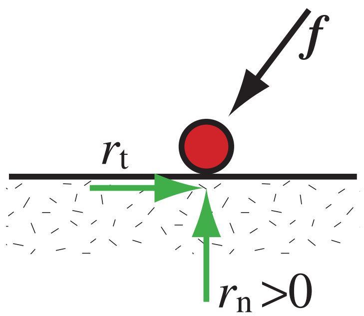

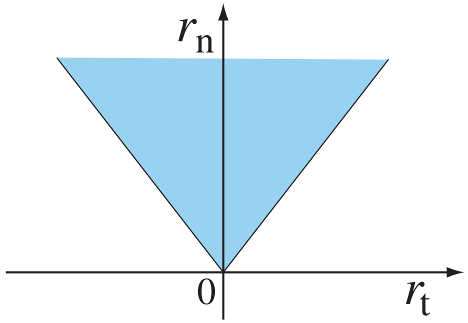

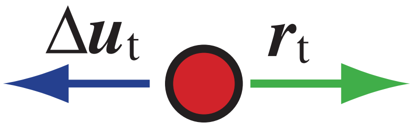

2 次錐相補性条件：

$$\mu r_{\mathrm{n}} \geq \|\boldsymbol{r}_{\mathrm{t}}\|, \quad \lambda_{\mathrm{n}} \geq \|\Delta \boldsymbol{u}_{\mathrm{t}}\|$$

$$\begin{pmatrix} \mu r_{\mathrm{n}} \\ \boldsymbol{r}_{\mathrm{t}} \end{pmatrix}^{\top} \begin{pmatrix} \lambda_{\mathrm{n}} \\ \Delta \boldsymbol{u}_{\mathrm{t}} \end{pmatrix} = 0$$

2次錐相補性問題

$$K\Delta \boldsymbol{u} = \boldsymbol{r} + \boldsymbol{f} \quad \text{：釣合式}$$

- 未知数：

$$\Delta \boldsymbol{u}, \quad \boldsymbol{r}$$

- 相補性条件：

$$\mu r_{\mathrm{n}i} \geq \|\boldsymbol{r}_{\mathrm{t}i}\|, \quad \lambda_{\mathrm{n}i} \geq \|\Delta \boldsymbol{u}_{\mathrm{t}i}\|,$$
$$(\mu r_{\mathrm{n}i}, \boldsymbol{r}_{\mathrm{t}i}) \cdot (\lambda_{\mathrm{n}i}, \Delta \boldsymbol{u}_{\mathrm{t}i}) = 0 \quad \text{：摩擦則}$$
$$g_i - \Delta u_{\mathrm{n}i} \geq 0, \quad r_{\mathrm{n}i} \geq 0,$$
$$(g_i - \Delta u_{\mathrm{n}i}) r_{\mathrm{n}i} = 0 \quad \text{：非貫通条件}$$

平滑化法

$$\boldsymbol{K}\Delta\boldsymbol{u} = \boldsymbol{r} + \boldsymbol{f} \quad \& \quad \textbf{(相補性条件)}$$

- **相補性条件:**

$$x \geq 0, \quad y \geq 0, \quad xy = 0$$

平滑化法

$$\boldsymbol{K}\Delta\boldsymbol{u} = \boldsymbol{r} + \boldsymbol{f} \quad \& \quad \textbf{(相補性条件)}$$

- **相補性条件**:

$$x \geq 0, \quad y \geq 0, \quad xy = 0$$

- **再定式化** (Fischer–Burminster **関数**):

$$\phi(x, y) := x + y - \sqrt{x^2 + y^2} = 0$$

平滑化法

$$\boldsymbol{K}\Delta\boldsymbol{u} = \boldsymbol{r} + \boldsymbol{f} \quad \& \quad$$ **(相補性条件)**

- **相補性条件**:

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- **再定式化** (Fischer–Burminster **関数**):

$$\phi(x, y) := x + y - \sqrt{x^2 + y^2} = 0$$

- **平滑化** [Kanzow 96]:

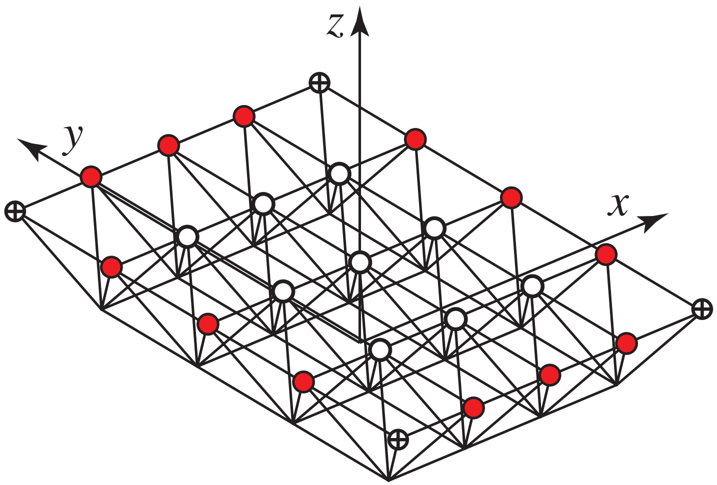
$$\phi_\epsilon(x, y) := x + y - \sqrt{x^2 + y^2 + \epsilon^2} = 0, \quad \epsilon > 0$$

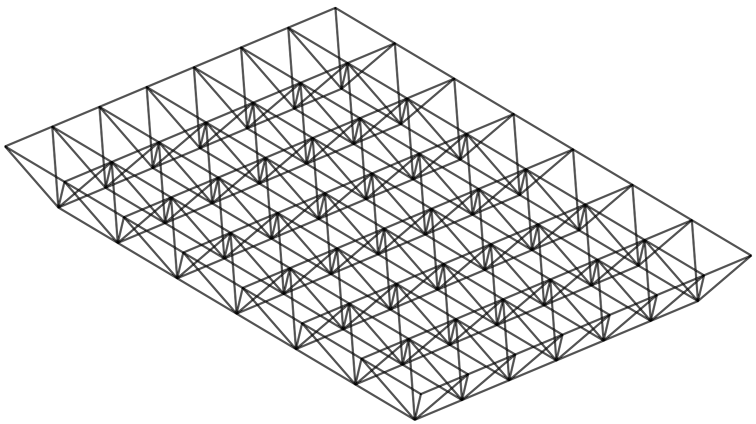
- 2 **次錐相補性問題に対する平滑化法** :

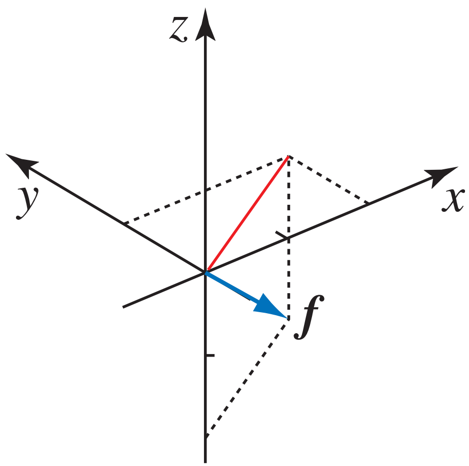
  [Hayashi–Yamashita–Fukushima 05]

例題

- **剛床上のトラス** $(\mathcal{U}_{\mathrm{ad}} = \{(x,y,z) | z \geq 0\})$







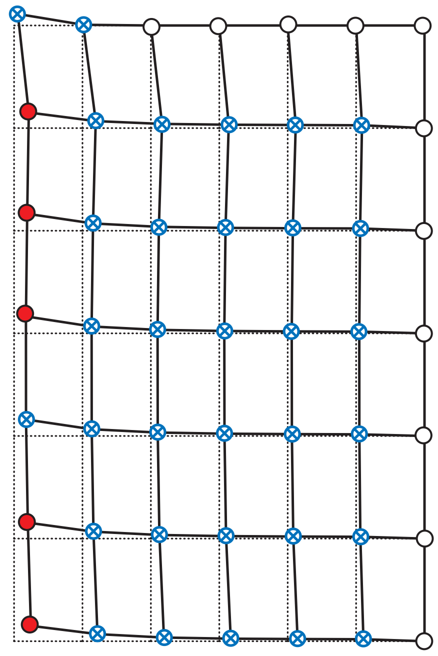
外力

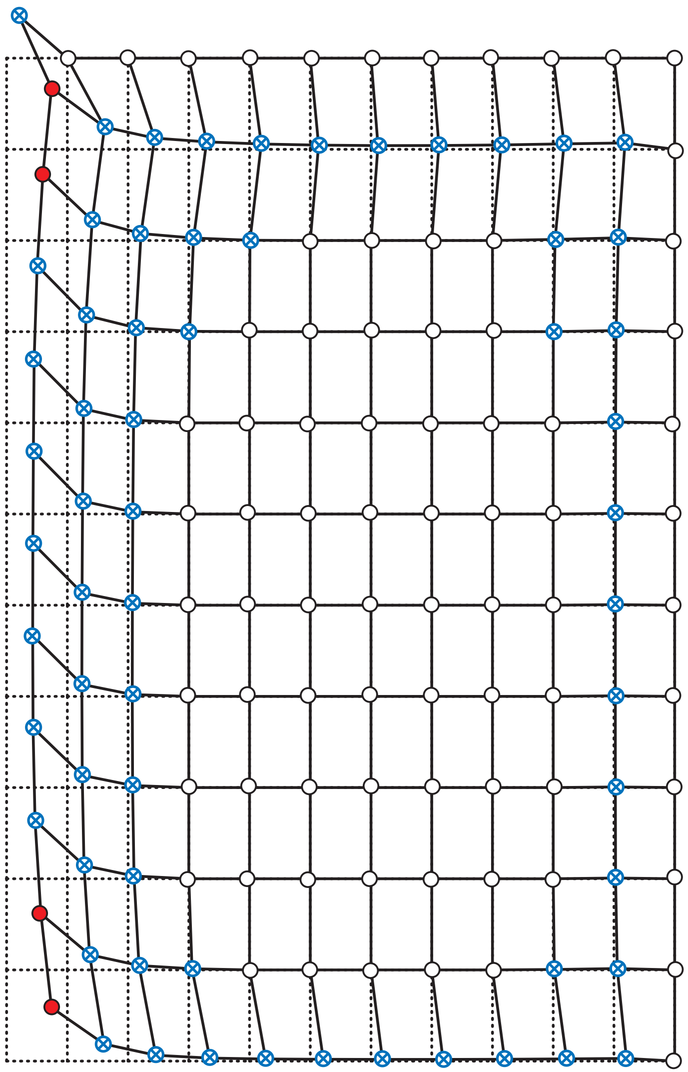
接触候補の変位 $(\mu = 1.5)$

○ - stuck

⊗ - slip

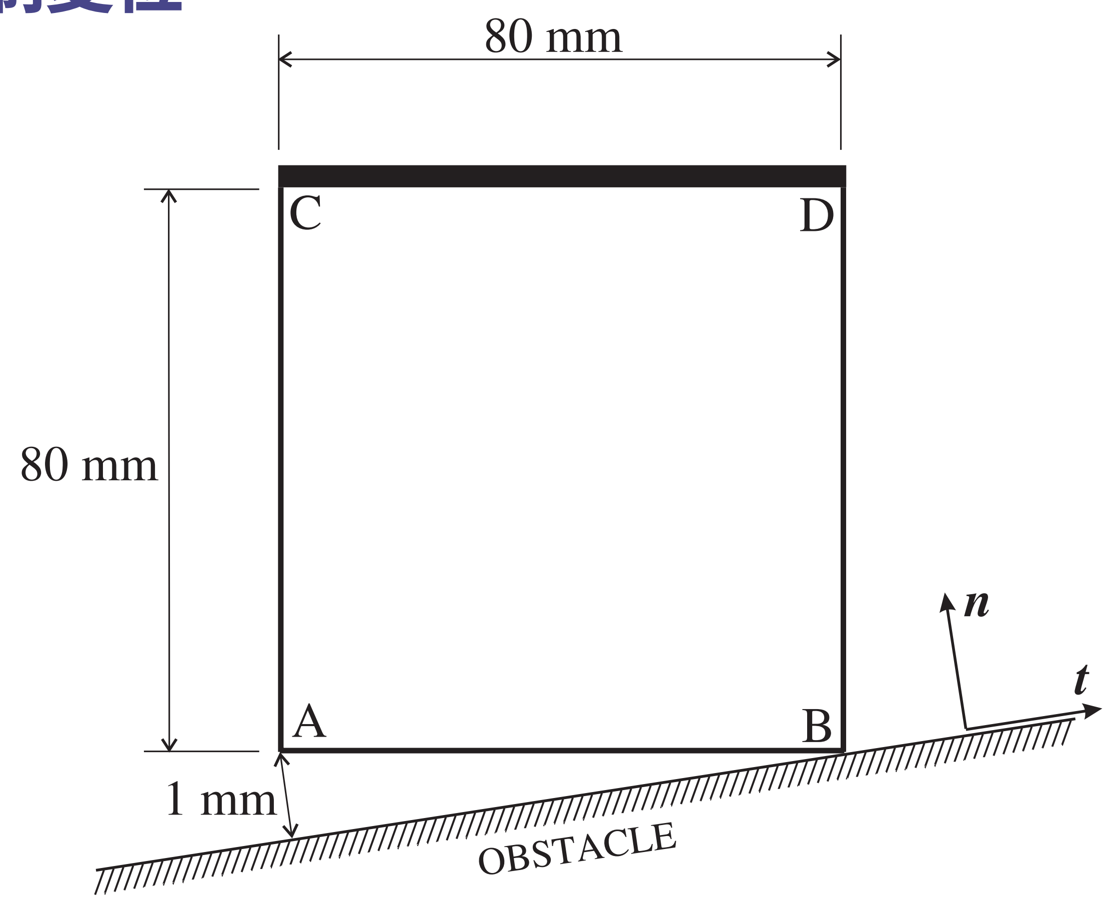
● - free





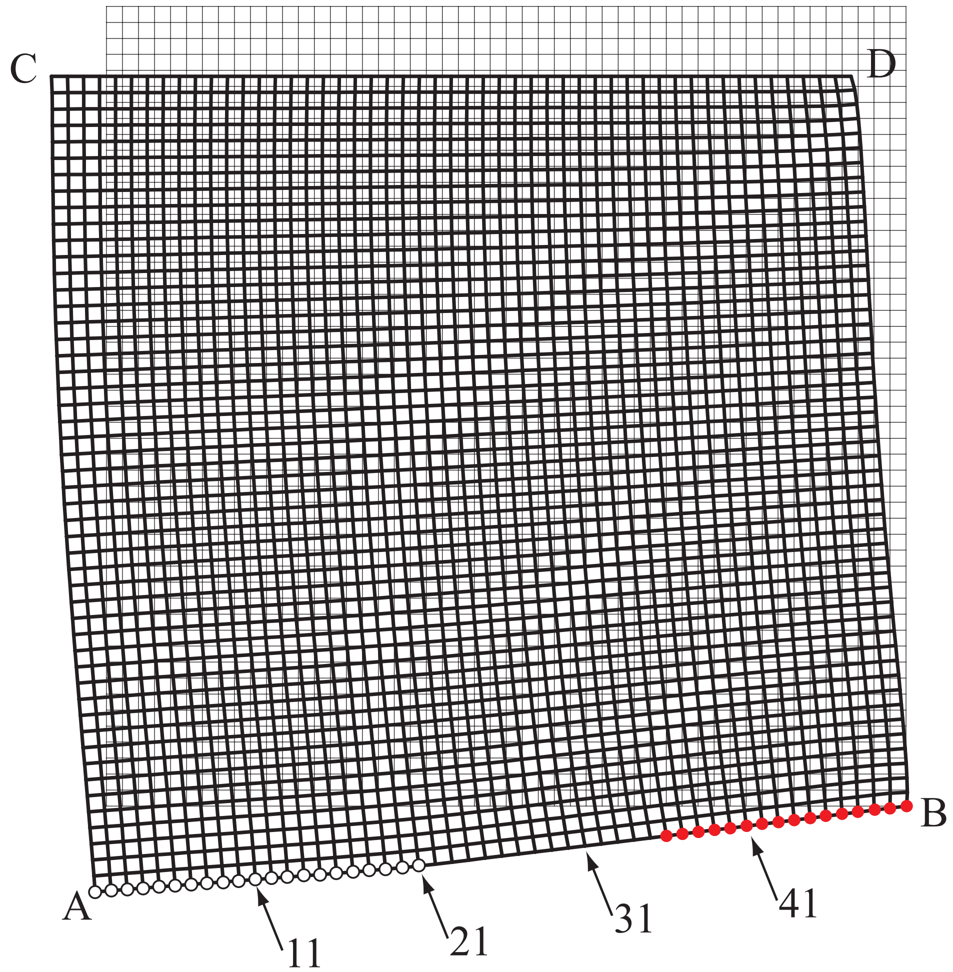
弾性体

- $E = 5$ MPa, $\nu = 0.4$, $\mu = 0.5$
- CD に強制変位



弾性体の釣合形状

- ○ - stuck
- ● - free



弾性体の釣合形状

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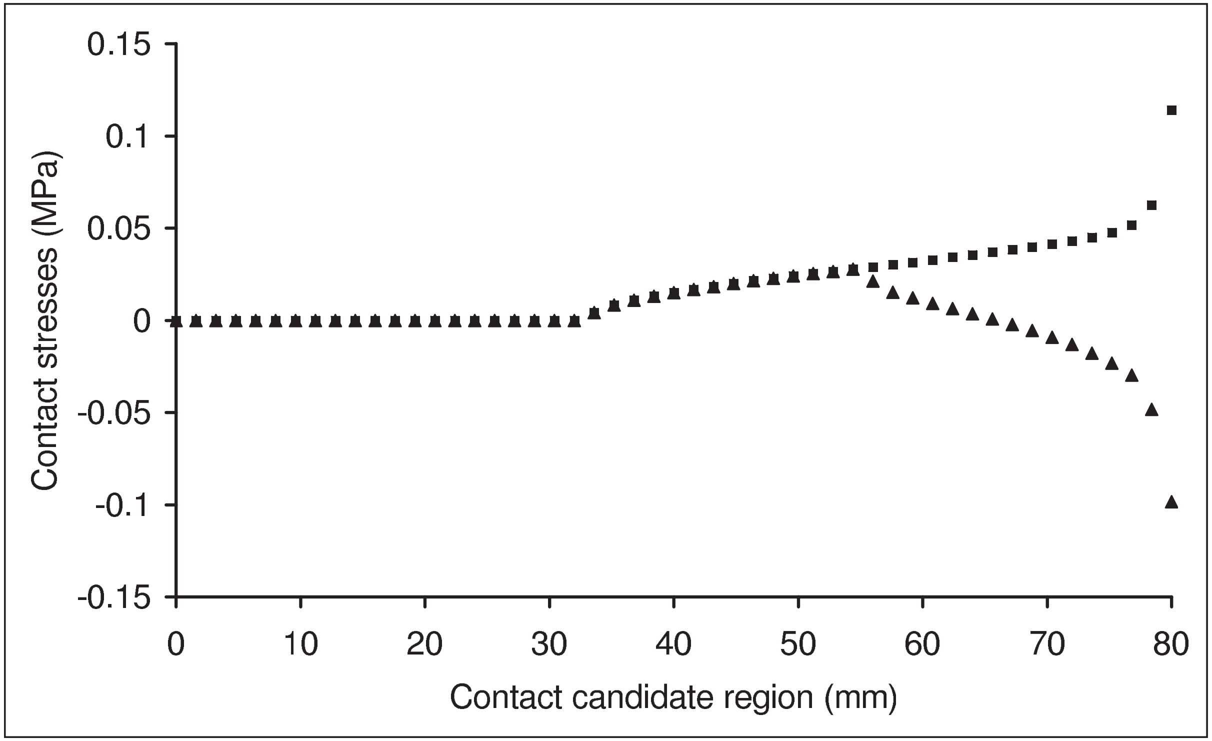
● - free



■ : $r_{\mathrm{n}i}$ ▲ : $r_{\mathrm{t}i}/\mu$

結論

- 準静的, 微小変形, 線形弾性
- Coulomb 摩擦, 剛体との接触
- 2 次錐相補性問題
  - 未知数:
    - 変位増分, 反力, 補助変数
  - 2 つの 2 次錐制約
  - 双線形な相補性条件
    - 2 次錐以外は微分可能
- 平滑化法 [Hayashi *et al.* 05]
- アルゴリズムの理論的な収束性
- 接触候補の節点に関する縮約