Exploring Frame Structures with Negative Poisson's Ratio via Mixed Integer Programming

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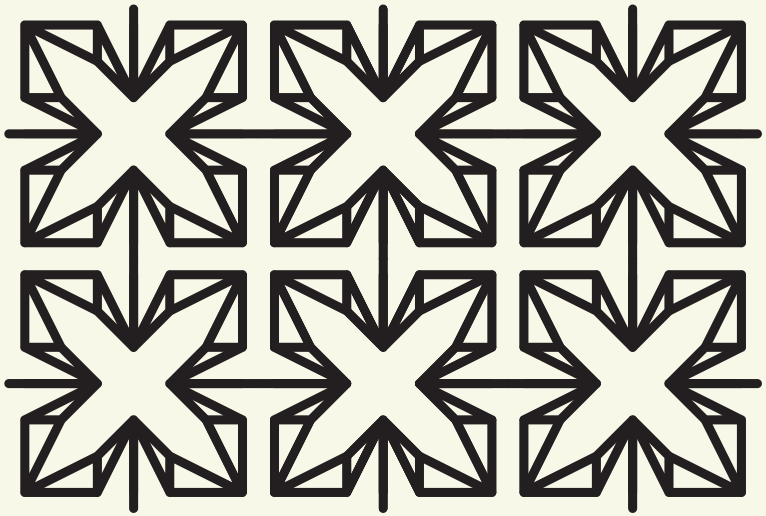
July 21, 2014 (WCCM XI, ECCM V, ECFD VI)

structures with negative Poisson's ratio

- ...expand transversely when stretched longitudinally.



mixed-integer programming

- m-i linear prog.:

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{r}^{\mathrm{T}}\boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{g}_i^{\mathrm{T}}\boldsymbol{y} \geq b_i \quad (i = 1, \dots, m), \\ & \boldsymbol{x} \in \{0, 1\}^n, \quad \boldsymbol{y} \in \mathbb{R}^l \end{aligned}$$

- “mixed”:
  - x_j : integer (discrete) variable
  - y_l : real (continuous) variable
- replace $\boldsymbol{x} \in \{0, 1\}^n$ with $\boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{1}$
  $\rightarrow$ linear prog. relaxation
- can be solved with, e.g., a branch-and-bound method

materials with NPR (= auxetic materials)

- naturally occurred materials
  - cadmium [Li '76]
  - single crystal of arsenic [Gunton & Saunders '72]
  - layered ceramics [Song, Zhou, Xu, Xu, & Bai '08]
- artificial materials
  - polymer foam [Lakes '87]
  - re-entrant structure [Friis, Lakes, & Park '88] [Evans, Alderson, & Christian '95]

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- artificial materials
  - polymer foam [Lakes '87]
  - re-entrant structure [Friis, Lakes, & Park '88] [Evans, Alderson, & Christian '95]
- (possible) applications
  - tunable filters [Alderson *et al.* '00]
  - fasteners [Choi & Lakes '91]
  - artificial intervertebra discs [Martz, Lakes, Goel, & Park '05]

optimization to achieve NPR

- existing methods:
  - truss model [Sigmund '94]
  - continuum & homogenization method [Larsen, Sigmund, & Bouwstra '97] [Schwerdtfeger *et al.* '11]
  - continuum & genetic alg. [Matsuoka, Yamamoto, & Takahara '01]

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- Local stress constraints were not considered.
- Post-processing before manufacturing: gray areas & hinges.

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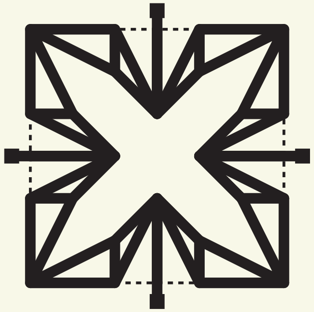
- our method:
  - periodic frame structure
  - stress constraints & pre-determined beam sections

  $\rightarrow$ no hinges, no thin members



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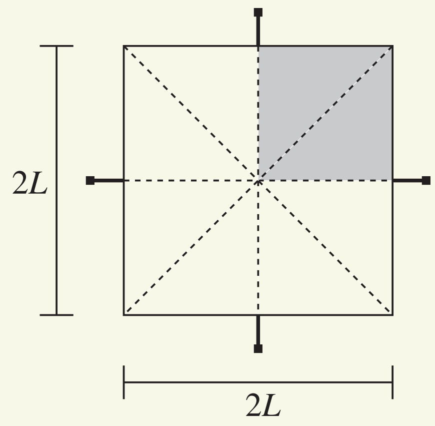
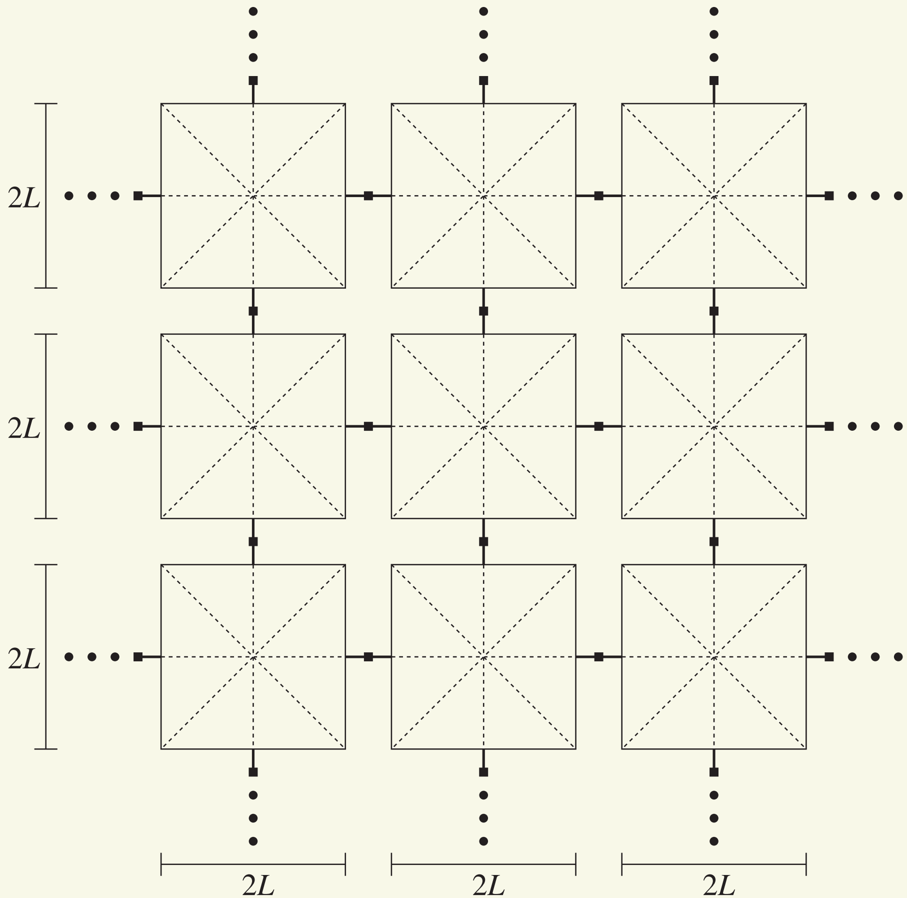
- our method:
  - periodic frame structure
  - stress constraints & pre-determined beam sections $\rightarrow$ no hinges, no thin members
  - $\rightarrow$ manufacturability (no post-processing)
  - $\rightarrow$ global optim.
    - an idea — MILP for truss [Rasmussen & Stolpe '08] [K. & Guo '10]

problem setting

- assume periodicity & symmetry
- unit cell: planar frame structure
- design variables: sections of members



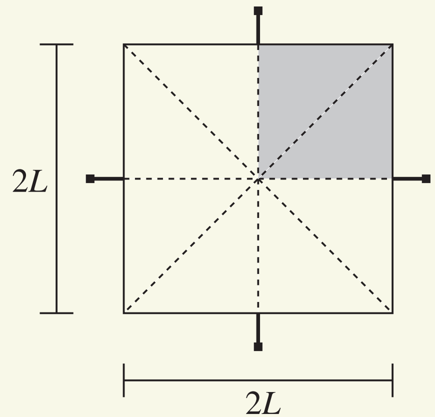
unit cell



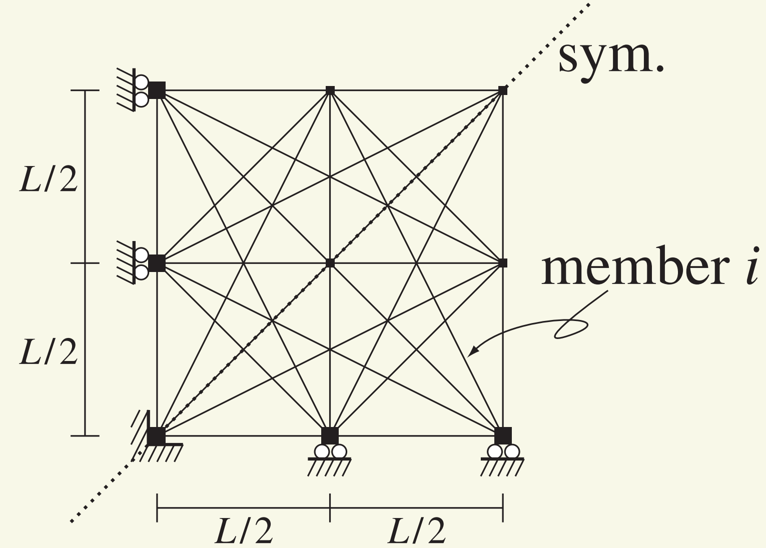
periodicity

problem setting

- assume periodicity & symmetry
- unit cell: planar frame structure
- design variables: sections of members
  - x_i : integer variable
    - $x_i = 1 \Rightarrow$ Member i has pre-determined section.
    - $x_i = 0 \Rightarrow$ Member i is removed.



unit cell



design domain

problem setting

- assume periodicity & symmetry
- unit cell: planar frame structure
- design variables: sections of members
  - more general setting:

    catalog $= \{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_P\}$

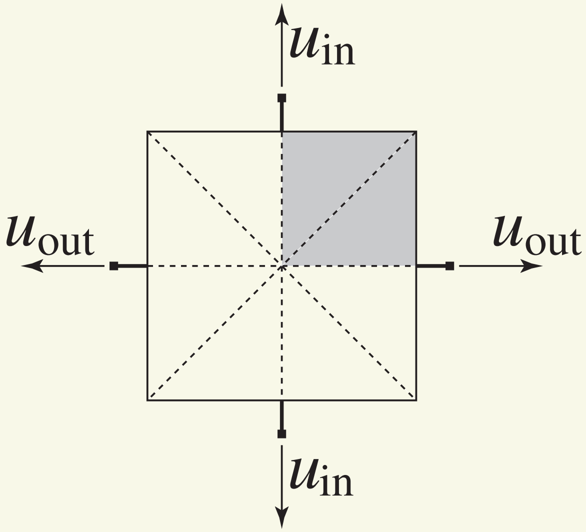
    - $x_{ip} = 1 \Rightarrow$ Member i has pre-determined section $\bar{a}_p$.
    - $x_{i1} = \cdots = x_{iP} = 0 \Rightarrow$ Member i is removed.

$$\boxed{\sum_p x_{ip} \leq 1}$$

    - (section) $= \sum_p x_{ip} \bar{a}_p$

optimization problem

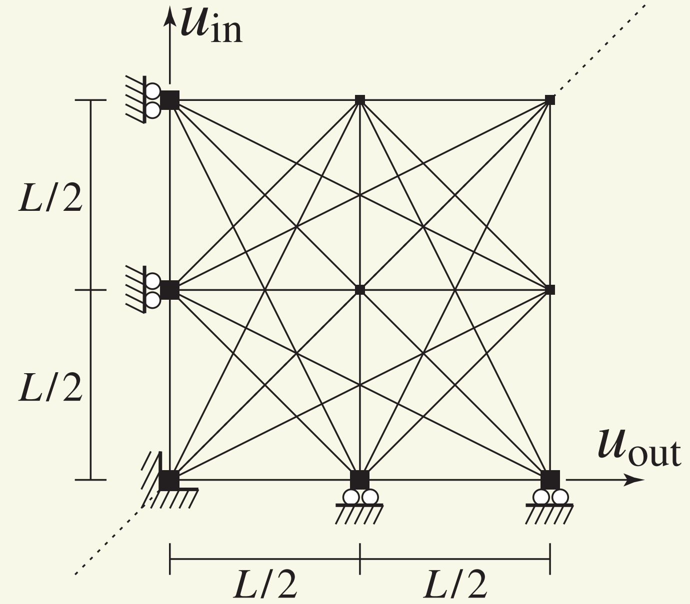
$$\begin{array}{ll} \max & u_{\rm out} \\ \text{s. t.} & \text{equilibrium eq.} \\ & \text{specifying } u_{\rm in} \\ & \text{stress constraints} \\ & \text{avoiding member intersection} \end{array}$$



- $\rightarrow$ can be reduced to m-i linear prog.

fictitious boundary cond.

- max. u_{out}
  - s. t. u_{in} is specified.
- "null structure" is optimal
  $\rightarrow u_{\mathrm{out}} = +\infty$



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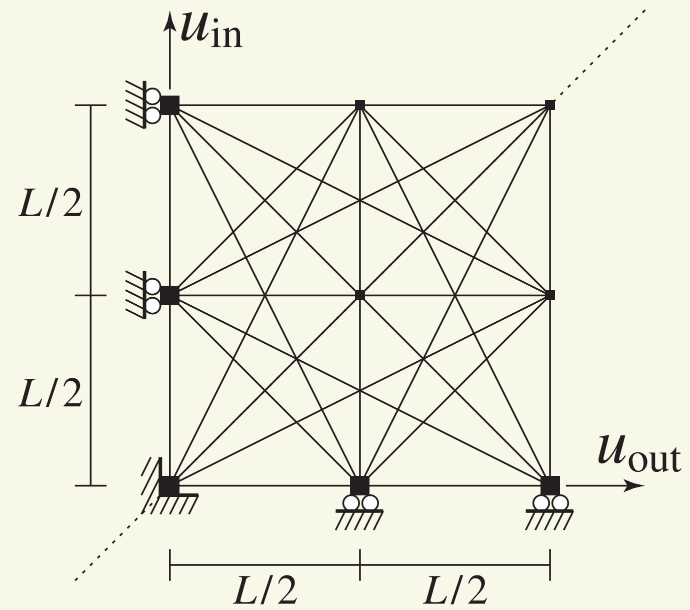

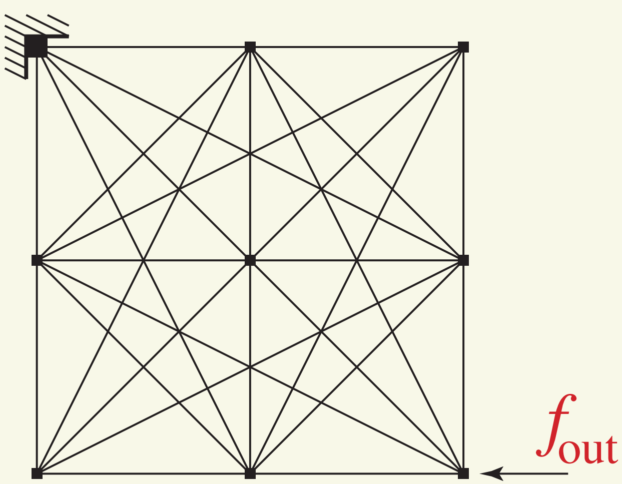

- fictitious cond.:
  - fix node "in"
  - apply f_{out} at node "out"
  - require "∃ internal forces satisfying force-balance eq."

reduction to MIP (1)

- equilibrium eq.:

$$K\boldsymbol{u} = \boldsymbol{f}$$

- stiffness matrix:

$$K = \sum_{i=1}^{m} \sum_{j=1}^{3} k_{ij} \boldsymbol{b}_{ij} \boldsymbol{b}_{ij}^{\mathrm{T}} \qquad (\boldsymbol{b}_{ij} \text{ : const. vec.})$$

- member stiffnesses:

$$k_{ij} = \bar{k}_{ij} x_i \qquad (\bar{k}_{ij} \text{ : const.})$$

- integer variable:

$$x_i = \begin{cases} 1 & \text{if member } i \text{ exists} \\ 0 & \text{if member } i \text{ is removed} \end{cases}$$

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- $\rightarrow$ Reformulate all constraints as linear constraints.

reduction to MIP (2)

- equil. eq. $K\boldsymbol{u} = \boldsymbol{f}$ $\quad\Leftrightarrow$

$$\sum_{i=1}^{m}\sum_{j=1}^{3} \bar{k}_{ij} v_{ij} \boldsymbol{b}_{ij} = \boldsymbol{f} \qquad \text{(force-balance)}$$

$$v_{ij} = \begin{cases} \boldsymbol{b}_{ij}^{\mathrm{T}}\boldsymbol{u} & \text{if } x_i = 1 \qquad (\diamondsuit) \\ 0 & \text{if } x_i = 0 \qquad (\clubsuit) \end{cases} \qquad \text{(compatibility)}$$

- stress constraints:

$$\frac{|q_i(\boldsymbol{u})|}{q_i^{\mathrm{y}}} + \frac{|m_i^{(e)}(\boldsymbol{u})|}{m_i^{\mathrm{y}}} \le 1 \qquad (\spadesuit)$$

reduction to MIP (2)

- equil. eq. $K\boldsymbol{u} = \boldsymbol{f}$ $\Leftrightarrow$

$$\sum_{i=1}^{m}\sum_{j=1}^{3} \bar{k}_{ij} v_{ij} \boldsymbol{b}_{ij} = \boldsymbol{f} \qquad \text{(force-balance)}$$

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- $(\clubsuit)$ & $(\spadesuit)$ $\Leftrightarrow$ $\displaystyle \frac{\bar{k}_{i1}}{q_i^{\mathrm{y}}}|v_{i1}| + \frac{l_i}{2}\frac{\bar{k}_{i2}}{m_i^{\mathrm{y}}}|v_{i2}| + \frac{\bar{k}_{i3}}{m_i^{\mathrm{y}}}|v_{i3}| \le x_i$

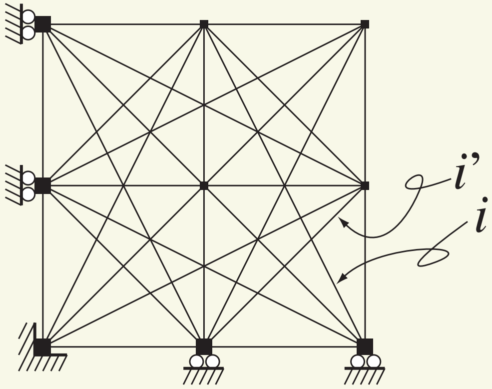
- $(\diamondsuit)$ $\Leftrightarrow$ $|v_{ij} - \boldsymbol{b}_{ij}^{\mathrm{T}}\boldsymbol{u}| \le M(1 - x_i)$ $\qquad$ ($M \gg 0$: const.)

goal: MIP formulation

$$
\begin{aligned}
\max \quad & u_{\rm out} \\
\text{s. t.} \quad & \sum_{i \in E} \sum_{j=1}^{3} \bar{k}_{ij} v_{ij} \boldsymbol{b}_{ij} = \boldsymbol{f}, \\
& |v_{ij} - \boldsymbol{b}_{ij}^{\rm T} \boldsymbol{u}| \le M(1 - x_i), & \forall j, \ \forall i, \\
& \frac{\bar{k}_{i1}}{q_i^{\rm y}} |v_{i1}| + \frac{l_i}{2} \frac{\bar{k}_{i2}}{m_i^{\rm y}} |v_{i2}| + \frac{\bar{k}_{i3}}{m_i^{\rm y}} |v_{i3}| \le x_{ip}, & \forall i, \\
& x_{ip} \in \{0, 1\}, & \forall i.
\end{aligned}
$$

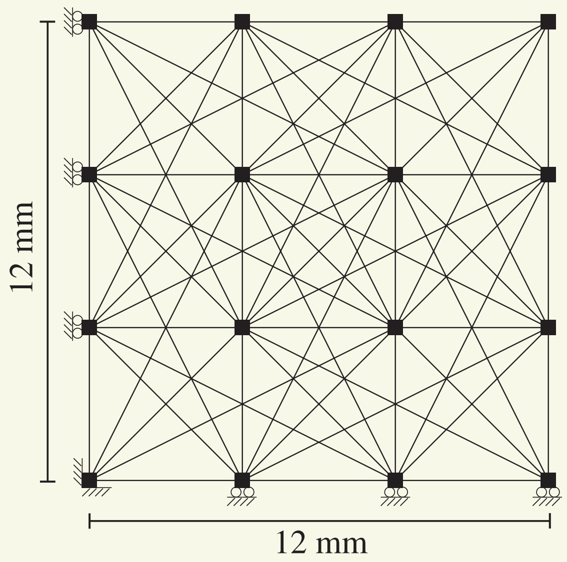
- avoiding member intersection:

$$x_i + x_{i'} \le 1$$



- only linear constraints (and integer constraints).

ex.) global optimization



- 66 candidate members
- Timoshenko beam element
- solver: CPLEX ver. 12.2

- beam cross-section
  - (width) $\times$ (thickness) $= 0.5 \times 0.5\,\mathrm{mm}$
  - (width) $\times$ (thickness) $= 1 \times 0.25\,\mathrm{mm}$

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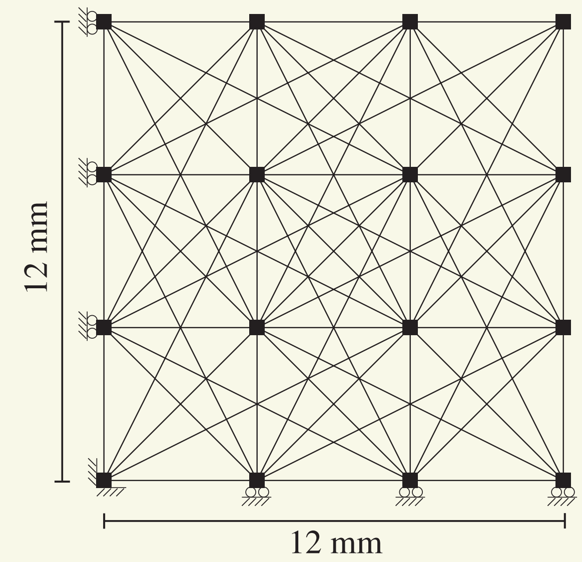

- beam cross-section
  - (width) $\times$ (thickness) $= 0.5 \times 0.5\,\mathrm{mm}$
    $\rightarrow \nu = -0.832887$
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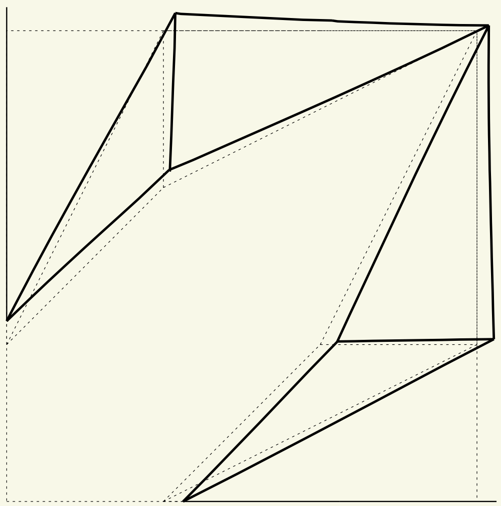
    $\rightarrow \nu = -0.752017$



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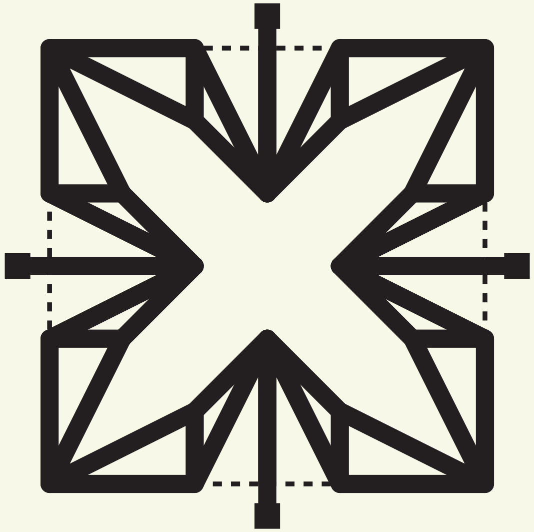
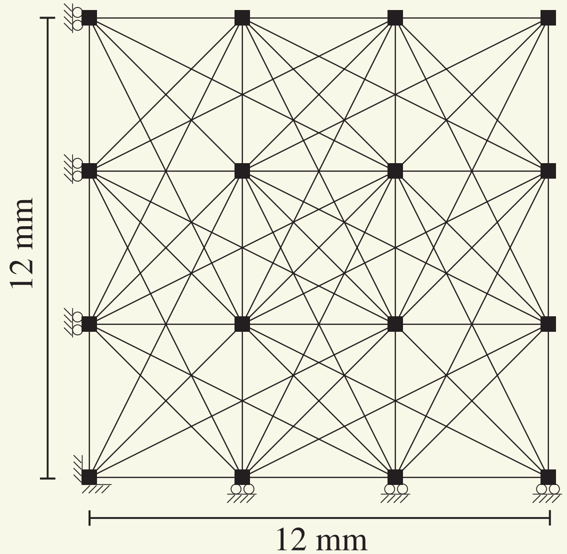


optimal base cell



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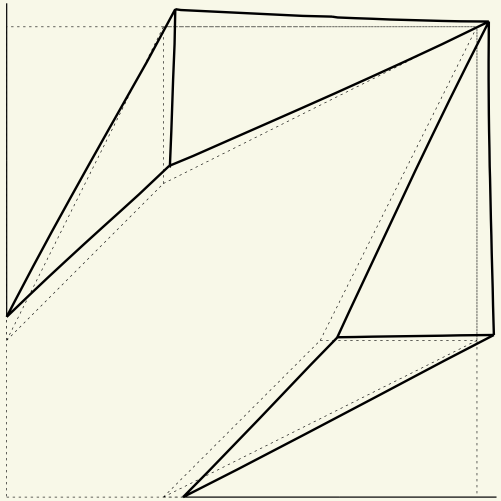
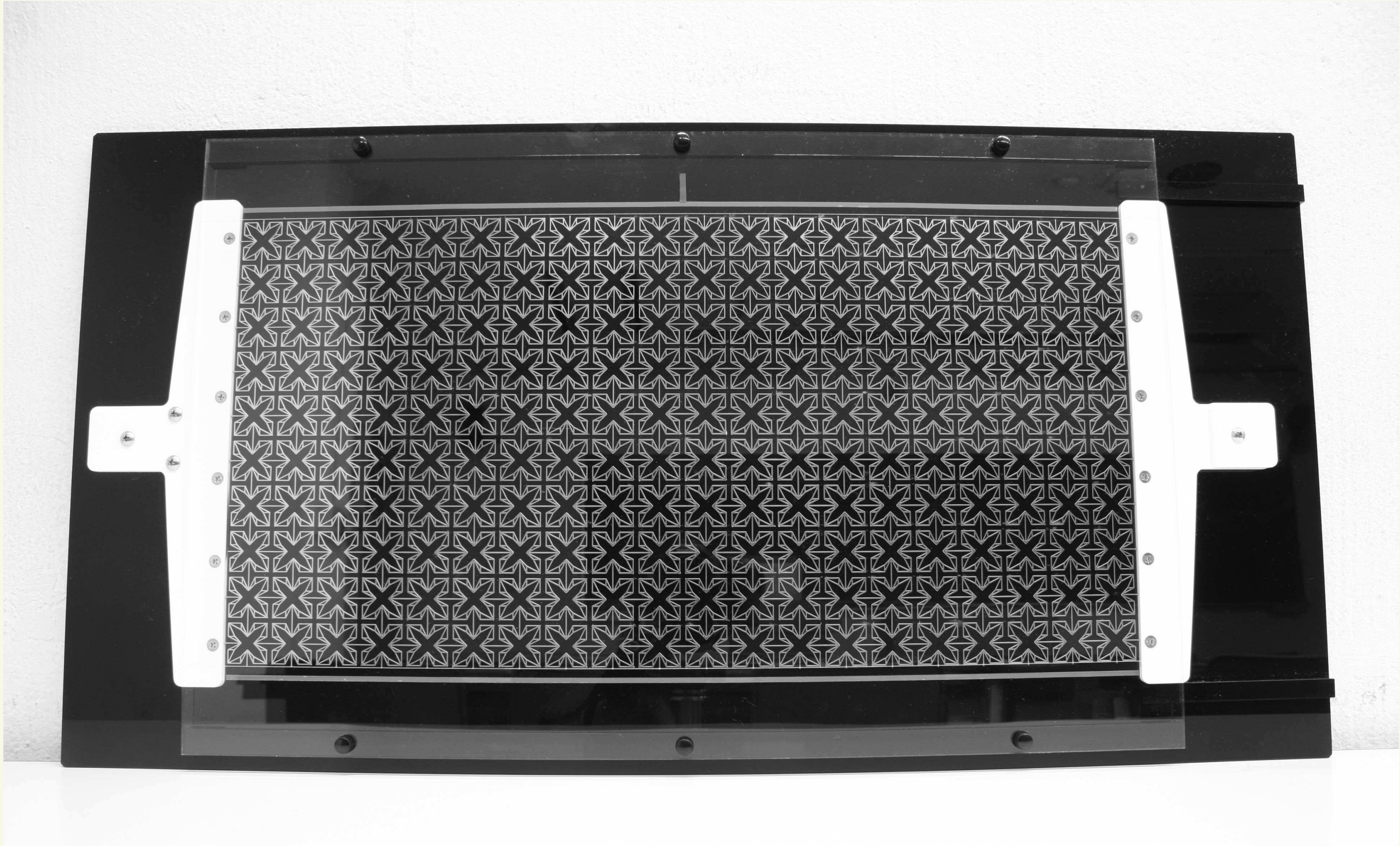


fabricated optimal structure

- fabricated by photo-etching
- stainless steel
  - thickness of beams: $0.5\,\mathrm{mm}$, width: $0.75\,\mathrm{mm}$



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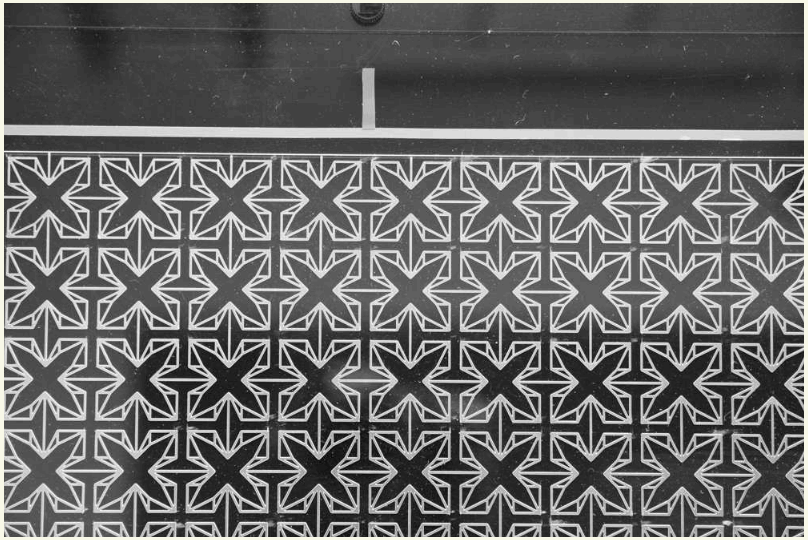
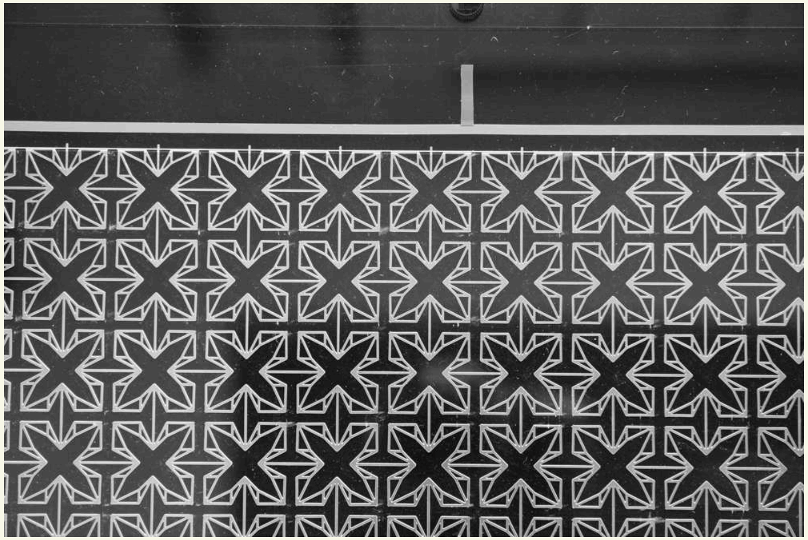


undeformed state



deformed state

local search: a heuristics

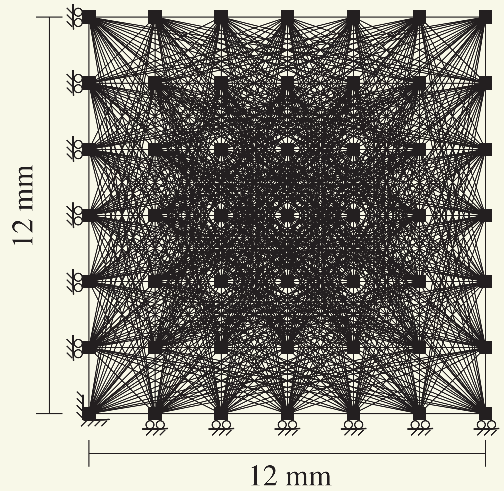
- MIP approach
  - global optim.
  - limitation of prob. size

- towards large probs.
  - local search with MIP [Stolpe & Stidsen '07] [Svanberg & Werme '07]
    - solve MIP within neighborhood $N(\boldsymbol{x}^*, r)$

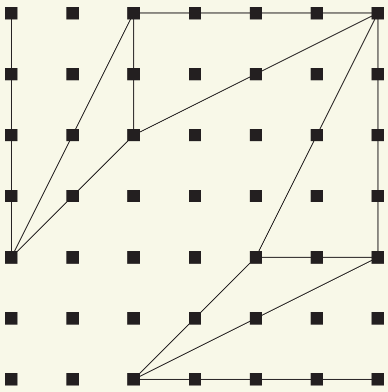
$$N(\boldsymbol{x}^*, r) = \left\{ \boldsymbol{x} \mid \sum_{i=1}^{m} |x_i - x_i^*| \leq r \right\}$$

    - r : radius
    - $\boldsymbol{x}^*$: incumbent solution

ex.) local search



candidate members

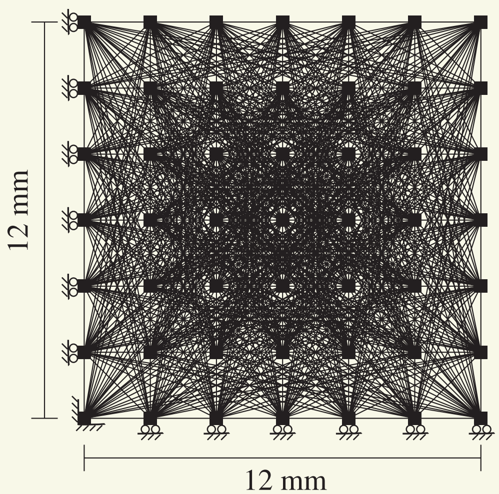


initial solution

$\nu = -0.832887$

- local search
  - 748 members
  - $r = 4$ (radius of neighborhood)
  - no guarantee of global optimality

ex.) local search



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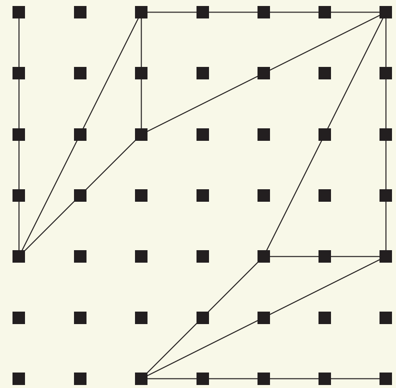

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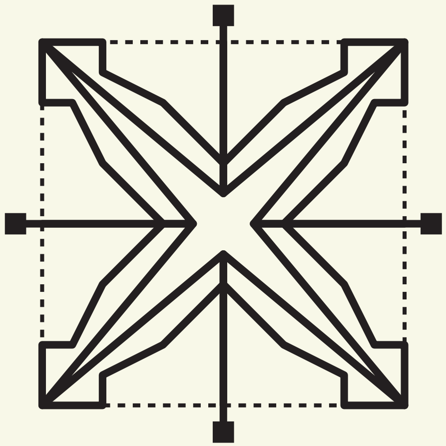
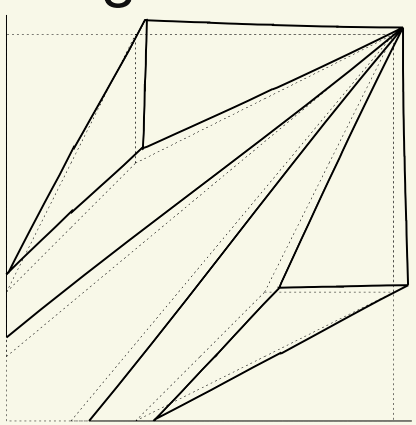


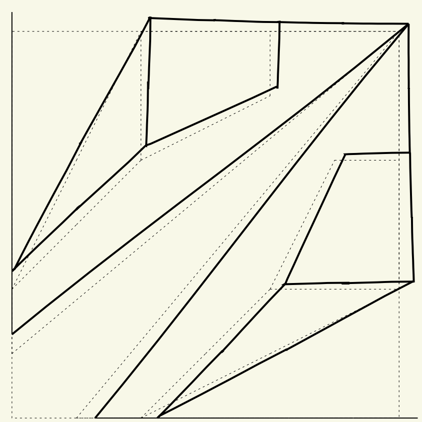
final design

$\nu = -0.969188$

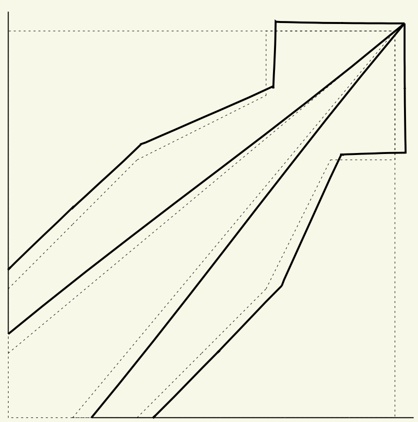
- convergence history:



1st step



2nd step



3rd step

conclusions

- design of counterintuitive structures
  - $\rightarrow$ Optimization might be a helpful tool.
- structures with negative Poisson's ratio
  - topology optimization of frame structures
    - max. the output displacement
    - mixed-integer programming
  - selection of member cross-sections $\leftarrow$ integer variables
    - from a catalog of available sections (incl. void)
  - stress constraints
  - no hinges, no thin members, no post-processing