Evaluation and Maximization of Robustness of Trusses by using Semidefinite Programming

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outline

- semidefinite programming (SDP)
- robustness function $\widehat{\alpha}(\boldsymbol{a})$
- how to compute $\widehat{\alpha}(\boldsymbol{a})$ for trusses
- MAX-$\widehat{\alpha}(\boldsymbol{a})$
  - find the truss design maximizing robustness function

semidefinite program, SDP

- convex, nonlinear
- includes LP, convex QP, etc.
- primal-dual interior-point methods [Kojima *et al.* 97]
  - can solve SDP in polynomial time
  - practical software
- application
  - eigenvalue optim. of trusses [Ohsaki *et al.* 99]
  - combinatorial optim. [Goemans & Williamson 95]
  - support vector machine [Lanckriet *et al.* 04]
  - robust LP [Ben-Tal & Nemirovski 02]

SDP

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \boldsymbol{C} - \sum_{i=1}^{m} \boldsymbol{A}_i y_i \succeq \boldsymbol{O} \end{aligned}$$

variables : $y_1, \dots, y_m$

coefficients : $b_1, \dots, b_m,$

$\boldsymbol{A}_1, \dots, \boldsymbol{A}_m, \ \boldsymbol{C} \in \mathcal{S}^n$

$\uparrow n \times n$ symmetric matrices

- $\boldsymbol{P} \succeq \boldsymbol{O} \quad \Longleftrightarrow \quad \boldsymbol{P}$ is positive semidefinite
  $\leftarrow$ nonlinear, convex constraint

uncertainty

- stochastic model
  - reliability design
- non-stochastic model
  - unknown-but-bounded

uncertainty

- stochastic model
  - reliability design
- non-stochastic model
  - unknown-but-bounded
  - convex model [Ben-Haim & Elishakoff 90]
    - robust truss optim. [Pantelides & Ganzerli 98]
  - robust LP, QP, SDP [Ben-Tal & Nemirovski 02]
    - robust truss optim. [Ben-Tal & Nemirovski 97]
  - sensitivity w.r.t. uncertain parameters
    - [Han & Kwak 04], etc...
  - robustness function [Ben-Haim 01]
    - measure of robustness

robustness function

- info-gap decision theory [Ben-Haim 01]
- $\widehat{\alpha}(\boldsymbol{a})$ — function of design variables $\boldsymbol{a}$
  - e.g. $\boldsymbol{a}$: member cross-sectional areas
- qualitative measure of robustness
  - $\widehat{\alpha}(\boldsymbol{a}_1) > \widehat{\alpha}(\boldsymbol{a}_2)$
    $\Longrightarrow \boldsymbol{a}_1$ is more robust than $\boldsymbol{a}_2$
- largest level of uncertainty
  - s.t. any constraint on mechanical performance cannot be violated

uncertain equilibrium eqs.

$$K(\mathsf{a})u = \mathsf{f} \qquad (\clubsuit)$$

uncertain a : $\quad \mathsf{a} = \widetilde{a} + \zeta_{\mathrm{a}}, \quad \alpha \geq \|\zeta_{\mathrm{a}}\|_\infty$

uncertain f : $\quad \mathsf{f} = \widetilde{f} + \zeta_{\mathrm{f}}, \quad \alpha \geq \|\zeta_{\mathrm{f}}\|_2$

$\widetilde{a}$, $\widetilde{f}$	nominal
ζ	unknown-but-bounded
$\alpha \geq 0$	‘level’ of uncertainty

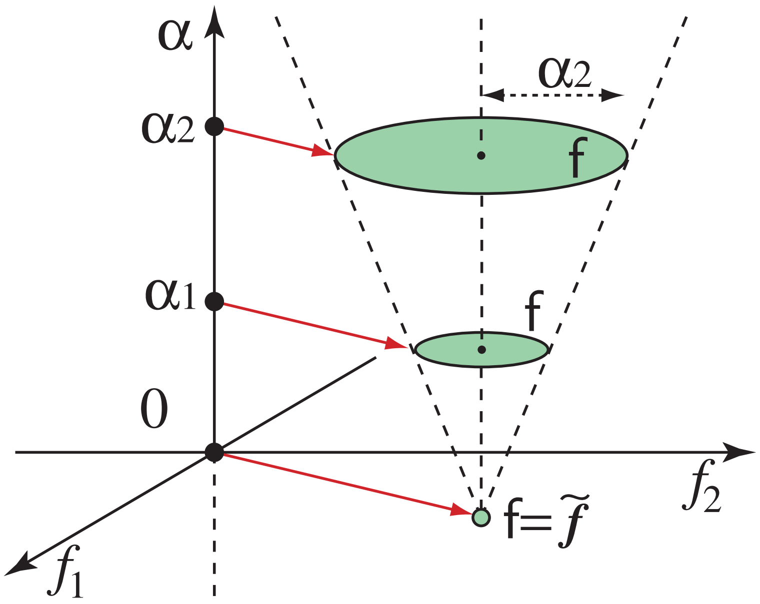
uncertain equilibrium eqs.

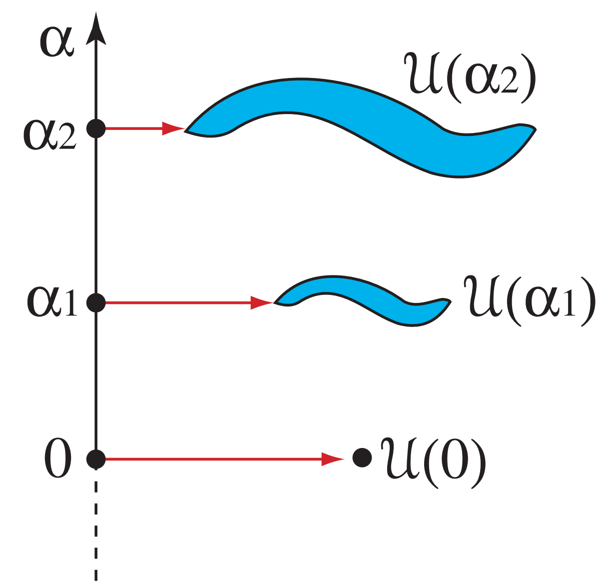
$$K(\mathsf{a})u = \mathsf{f} \qquad (\clubsuit)$$

uncertain a : $\quad \mathsf{a} = \tilde{a} + \zeta_{\mathrm{a}}, \quad \alpha \geq \|\zeta_{\mathrm{a}}\|_\infty$

uncertain f : $\quad \mathsf{f} = \tilde{f} + \zeta_{\mathrm{f}}, \quad \alpha \geq \|\zeta_{\mathrm{f}}\|_2$

- $\mathcal{U}(\alpha) \doteq$ set of u solving $(\clubsuit)$





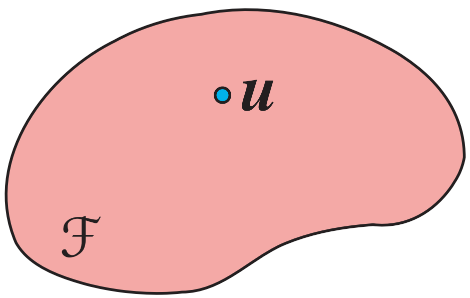
constraints

nominal constraints

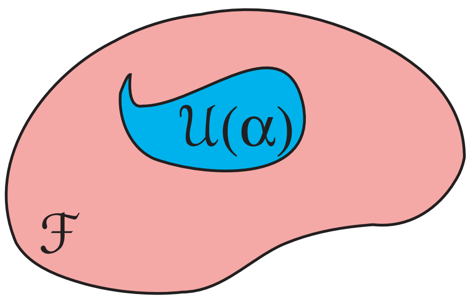
$$\boldsymbol{u} \in \mathcal{F}, \quad \boldsymbol{u} \text{ solves equilibrium eqs.}$$

robust constraints

$$\mathcal{U}(\alpha) \subseteq \mathcal{F}$$



$\boldsymbol{u} \in \mathcal{F}$, $\boldsymbol{K}\boldsymbol{u} = \widetilde{\boldsymbol{f}}$

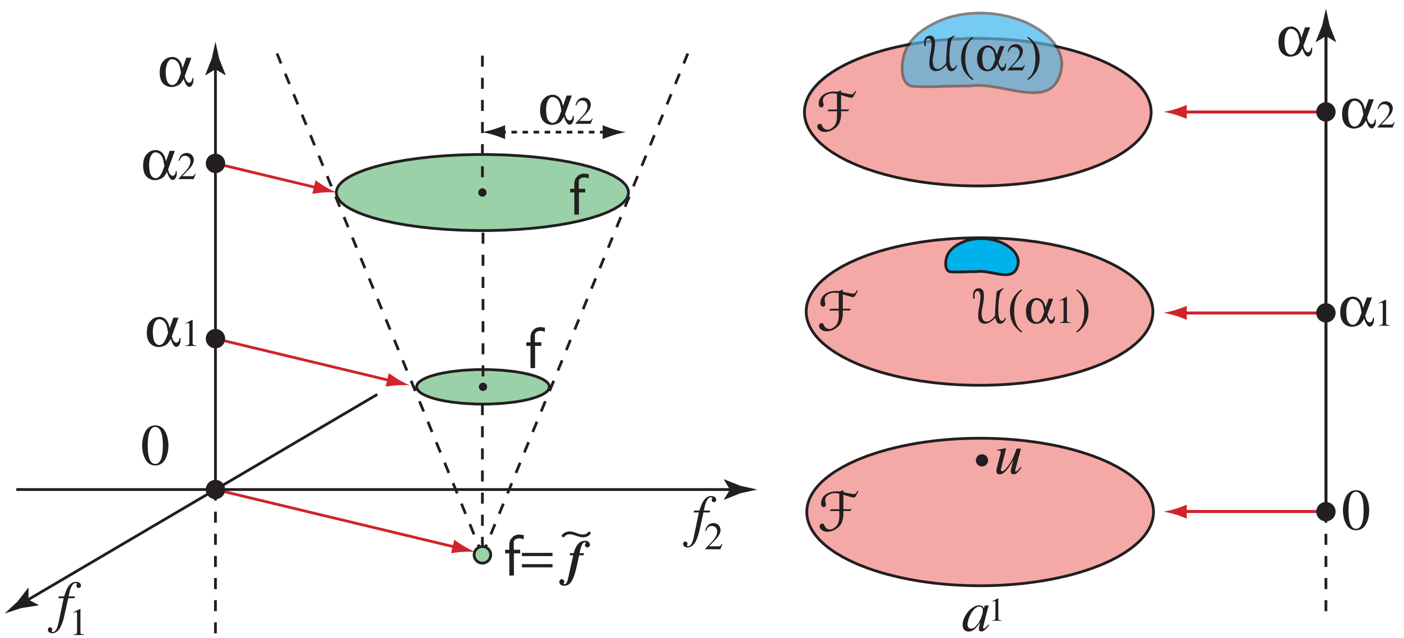


$\boldsymbol{u} \in \mathcal{F}$, $\forall \boldsymbol{u} \in \mathcal{U}(\alpha)$

robustness function $\widehat{\alpha}$

robust constraints

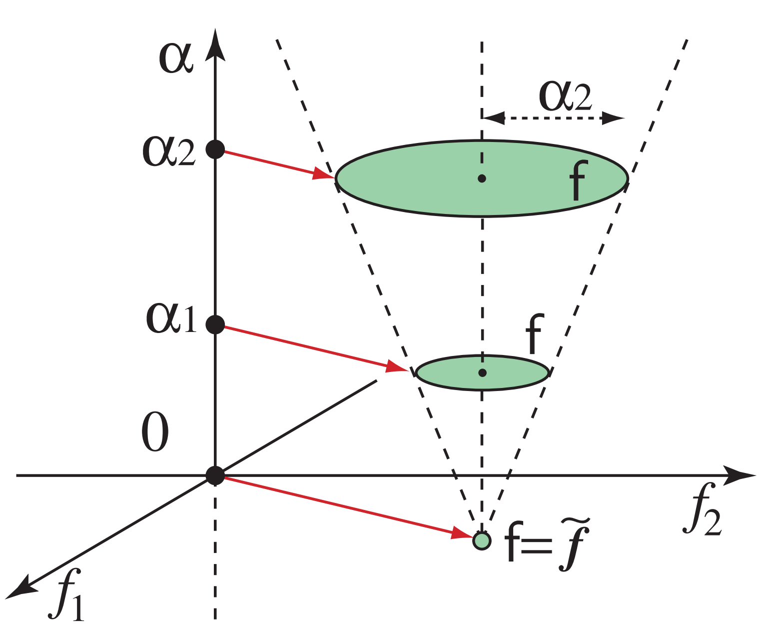
$$\mathcal{U}(\alpha) \subseteq \mathcal{F}$$

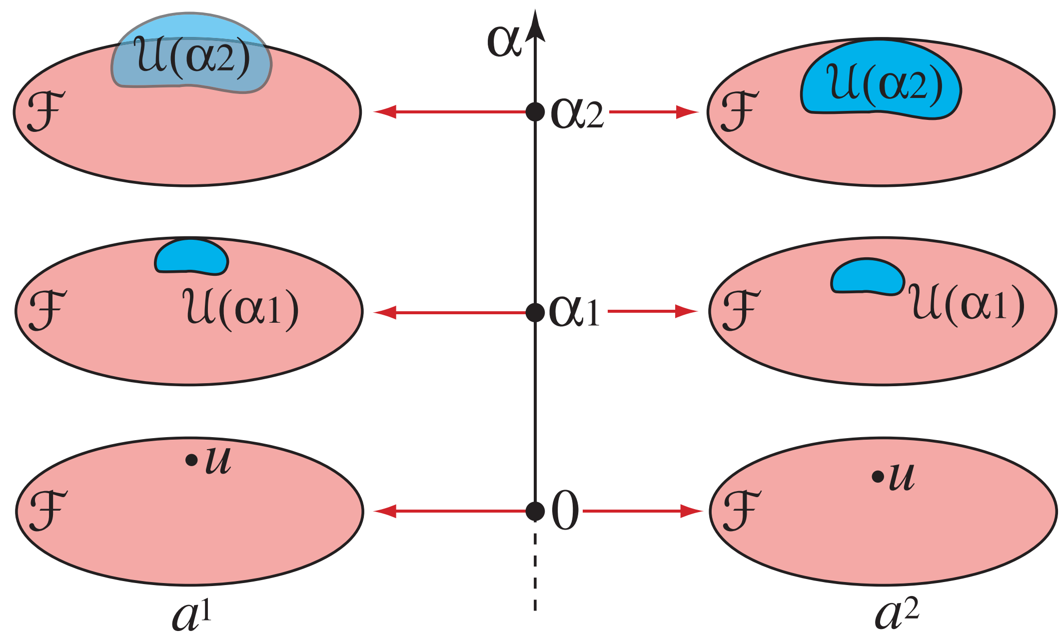


robustness function $\widehat{\alpha}$

robust constraints

$$\mathcal{U}(\alpha) \subseteq \mathcal{F}$$



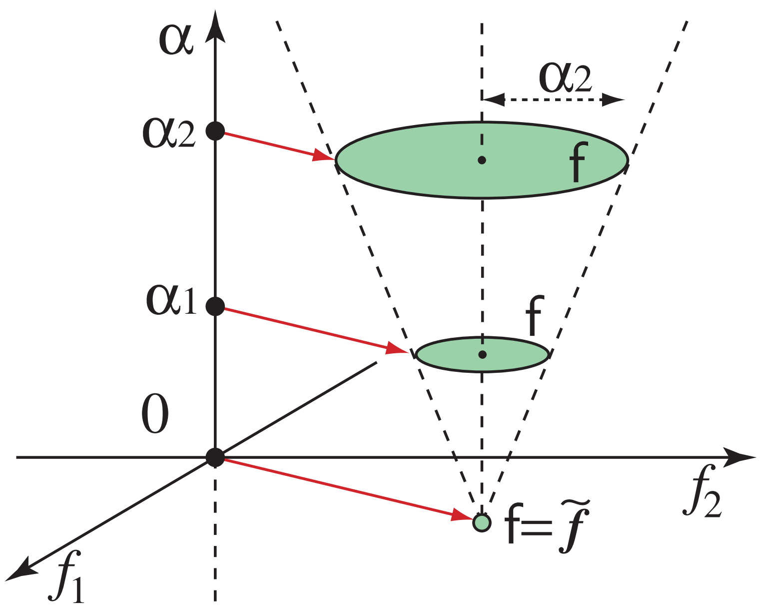


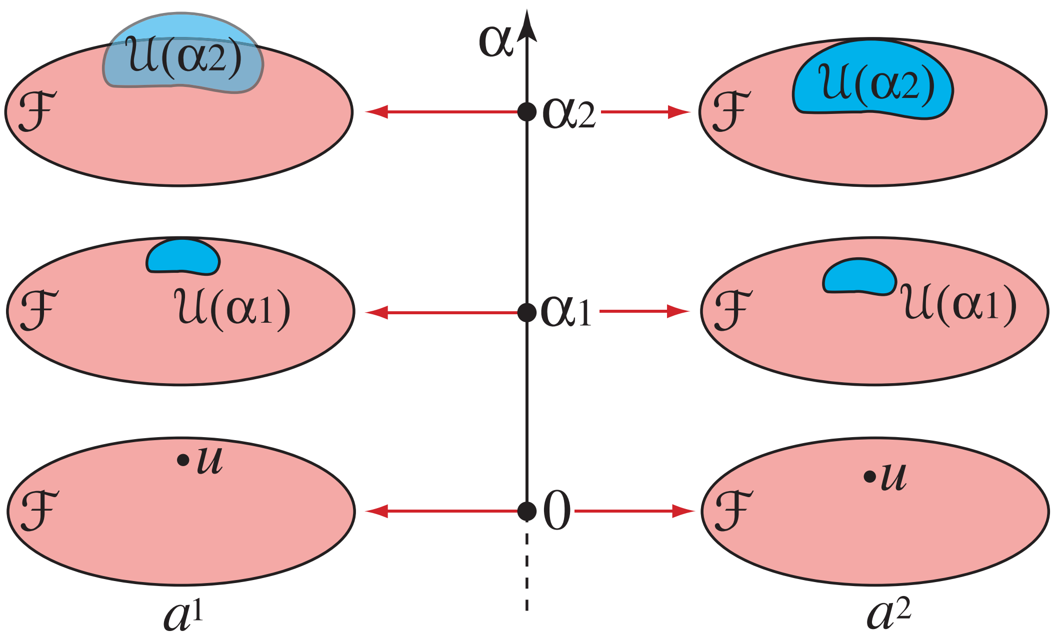
$\widehat{\alpha}(\boldsymbol{a}^1) = \alpha_1$ $\qquad$ $\widehat{\alpha}(\boldsymbol{a}^2) = \alpha_2$

robustness function $\widehat{\alpha}$

robust constraints

$$\mathcal{U}(\alpha) \subseteq \mathcal{F}$$





$\widehat{\alpha}(\boldsymbol{a}^1) = \alpha_1$ $\qquad$ $\widehat{\alpha}(\boldsymbol{a}^2) = \alpha_2$

- $\widehat{\alpha} \doteq$ max α s.t. $\boldsymbol{u} \in \mathcal{F}$ is always satisfied
- $\widehat{\alpha} = \max\{\alpha | \mathcal{U}(\alpha) \subseteq \mathcal{F}\}$

robustness function $\widehat{\alpha}$

$\mathcal{F}$ — feasible set

$$\mathcal{F} = \{\boldsymbol{u} \,|\, \boldsymbol{g}(\boldsymbol{u}) \leq \boldsymbol{0}\}$$

$g_i(\boldsymbol{u}) \quad \leftarrow \quad$ polynomial in $\boldsymbol{u}$

$\mathcal{U}(\alpha)$ — set of $\boldsymbol{u}$ solving uncertain equilibrium eqs.

$$\boldsymbol{u} \in \mathcal{U}(\alpha)$$

$$\Updownarrow$$

$$\boldsymbol{K}(\widetilde{\boldsymbol{a}} + \zeta_{\mathrm{a}})\boldsymbol{u} = \widetilde{\boldsymbol{f}} + \zeta_{\mathrm{f}}, \quad \alpha \geq \|\zeta_{\mathrm{a}}\|_{\infty}, \quad \alpha \geq \|\zeta_{\mathrm{f}}\|_2$$

$\widehat{\alpha}$ — robustness function

$$\widehat{\alpha} = \max\{\alpha | \mathcal{U}(\alpha) \subseteq \mathcal{F}\}$$

quadratic embedding

- $\mathcal{U}(\alpha)$ — set of $\boldsymbol{u}$ solving uncertain equilibrium eqs.

fact:

for any $\alpha \geq 0$, $\mathcal{U}(\alpha)$ can be expressed via some quadratic inequalities in $\boldsymbol{u}$, i.e., letting $\boldsymbol{\Omega}_l(\alpha) \in \mathcal{S}^{n+1}$,

$$\mathcal{U}(\alpha) = \left\{ \boldsymbol{u} \,\middle|\, \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}^{\top} \boldsymbol{\Omega}_l(\alpha) \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix} \geq 0,\ l = 1, \dots, m \right\}$$

- $\mathcal{F}$ — feasible set

fact:

$\mathcal{F}$ can be expressed via some quadratic inequalities in $\boldsymbol{u}$

- eliminate uncertain parameters ζ

$\mathcal{S}$-procedure + homogenization

quadratic inequalities

$$\mathcal{Q}_l = \left\{ \boldsymbol{u} \left| \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}^{\top} \boldsymbol{P}_l \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix} \geq 0 \right. \right\}, \quad \boldsymbol{P}_0, \boldsymbol{P}_1, \dots, \boldsymbol{P}_m \in \mathcal{S}^{n+1}$$

fact:

$$(\mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_m) \subseteq \mathcal{Q}_0$$

$$\Uparrow$$

$$\exists \tau_1, \dots, \tau_m \geq 0, \quad \boldsymbol{P}_0 - \sum_{l=1} \tau_l \boldsymbol{P}_l \succeq \boldsymbol{O}$$

$\mathcal{S}$-procedure + homogenization

quadratic inequalities

$$\mathcal{Q}_l = \left\{ \boldsymbol{u} \;\middle|\; \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix}^{\top} \boldsymbol{P}_l \begin{pmatrix} \boldsymbol{u} \\ 1 \end{pmatrix} \geq 0 \right\}, \qquad \boldsymbol{P}_0, \boldsymbol{P}_1, \dots, \boldsymbol{P}_m \in \mathcal{S}^{n+1}$$

fact (special case):

$$\mathcal{Q}_1 \subseteq \mathcal{Q}_0$$

$$\Updownarrow$$

$$\exists \tau_1 \geq 0, \quad \boldsymbol{P}_0 - \tau_1 \boldsymbol{P}_1 \succeq \boldsymbol{O}$$

SDP providing a lower bound of $\widehat{\alpha}$

$$\widehat{\alpha}(\widetilde{\boldsymbol{a}}) = \max \{ \alpha \,|\, \mathcal{U}(\alpha) \subseteq \mathcal{F} \}$$

$$\downarrow$$

$\mathcal{S}$-procedure

$$\downarrow$$

SDP in $(t, \boldsymbol{\rho})$:

$$(\widehat{\alpha}(\widetilde{\boldsymbol{a}}))^2 \geq \max \{ t \,|\, \overline{\boldsymbol{G}}(t, \boldsymbol{\rho}) \succeq \boldsymbol{O}, \ \boldsymbol{\rho} \geq \boldsymbol{0} \}$$

SDP providing a lower bound of $\widehat{\alpha}$

$$\widehat{\alpha}(\widetilde{\boldsymbol{a}}) = \max\{\alpha \,|\, \mathcal{U}(\alpha) \subseteq \mathcal{F}\}$$

$\downarrow$

$\mathcal{S}$-procedure

$\downarrow$

SDP in $(t, \boldsymbol{\rho})$:

$$(\widehat{\alpha}(\widetilde{\boldsymbol{a}}))^2 \geq \max\{t \,|\, \overline{\boldsymbol{G}}(t, \boldsymbol{\rho}) \succeq \boldsymbol{O},\ \ \boldsymbol{\rho} \geq \boldsymbol{0}\}$$

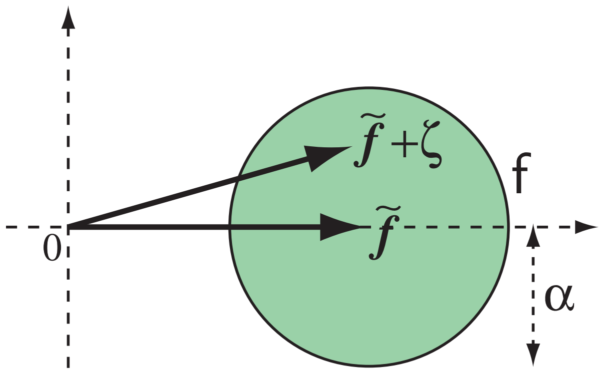
particularly, if only $\boldsymbol{f}$ is uncertain, then

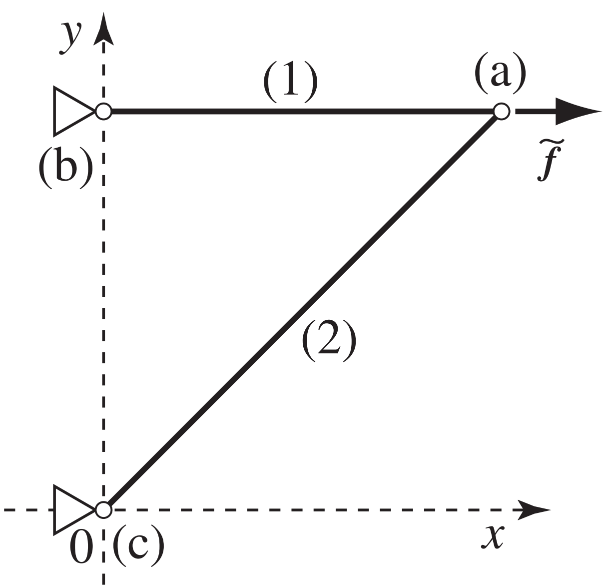
$$(\widehat{\alpha}(\boldsymbol{a}))^2 = \max\{t \,|\, \boldsymbol{G}(t, \boldsymbol{\rho}) \succeq \boldsymbol{O},\ \ \boldsymbol{\rho} \geq \boldsymbol{0}\}$$

- $\widehat{\alpha}$ is obtained by solving an SDP

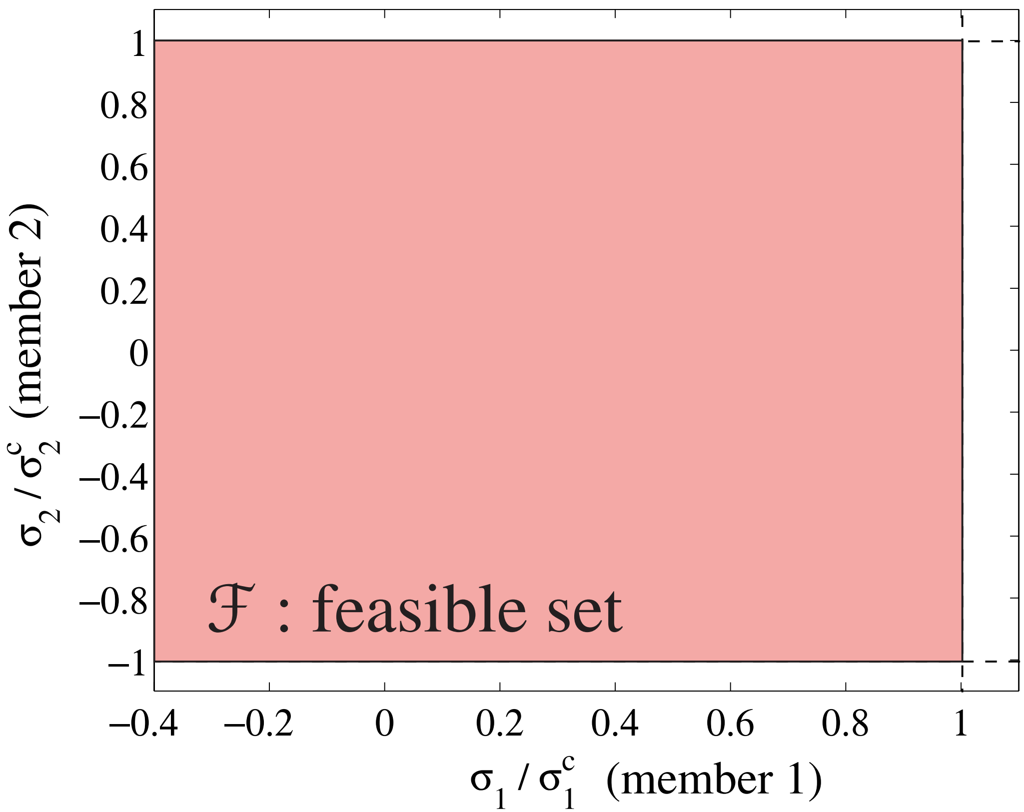
example (2-bar truss)

- $\mathbf{f} = \widetilde{f} + \zeta, \quad \alpha \geq \|\zeta\|_2$
- stress constraints





$\widehat{\alpha} = 69.297$ kN



stress states

example (2-bar truss)

- $\mathbf{f} = \widetilde{f} + \zeta, \quad \alpha \geq \|\zeta\|_2$
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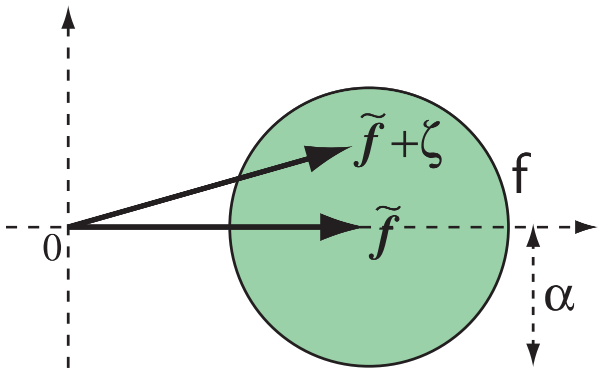

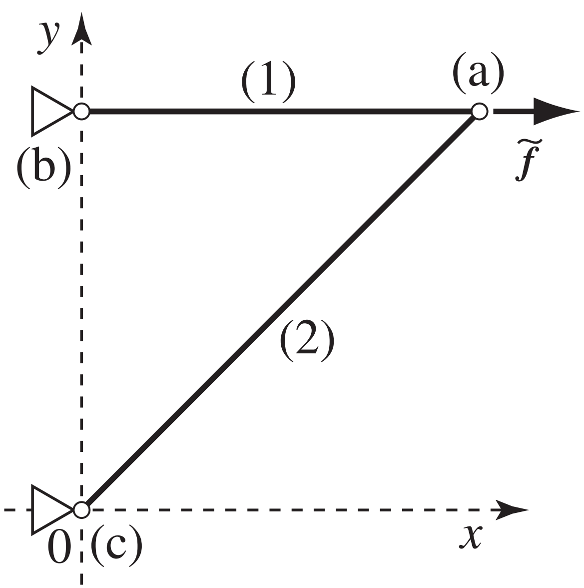

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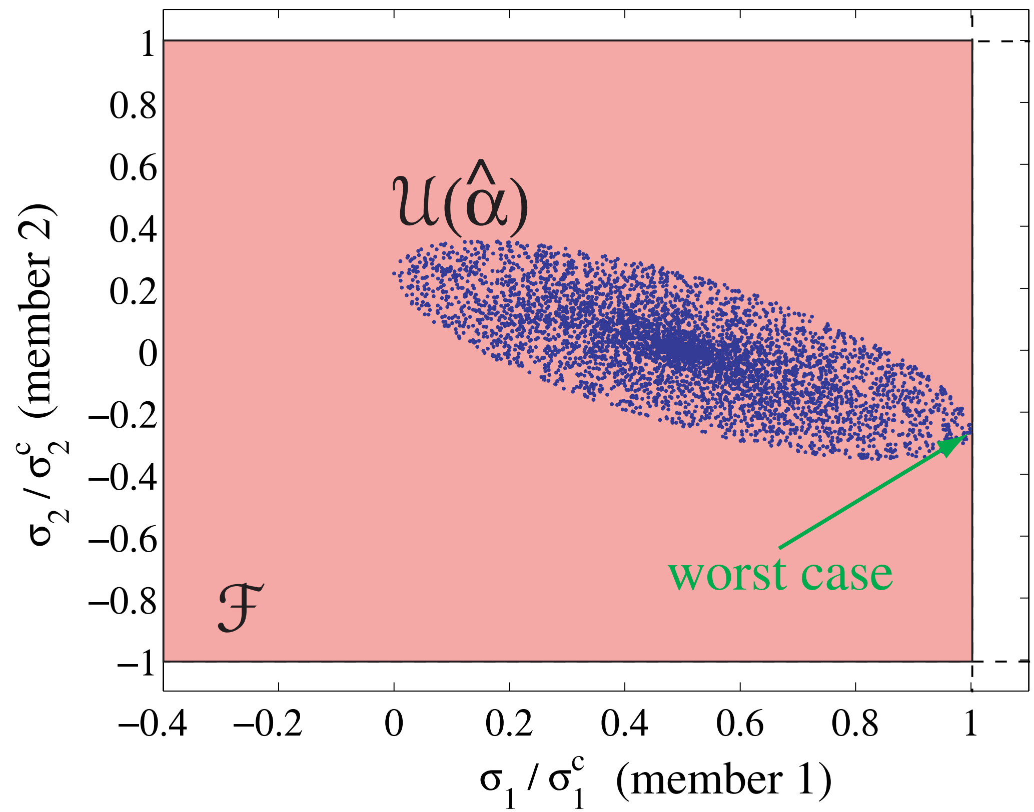
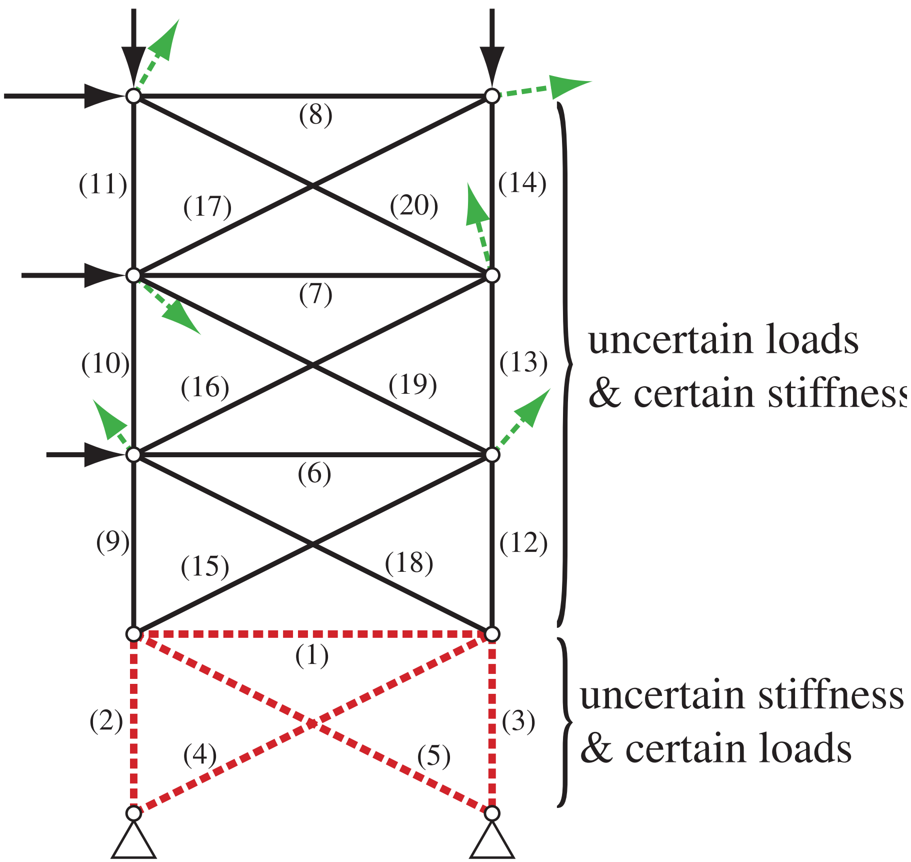


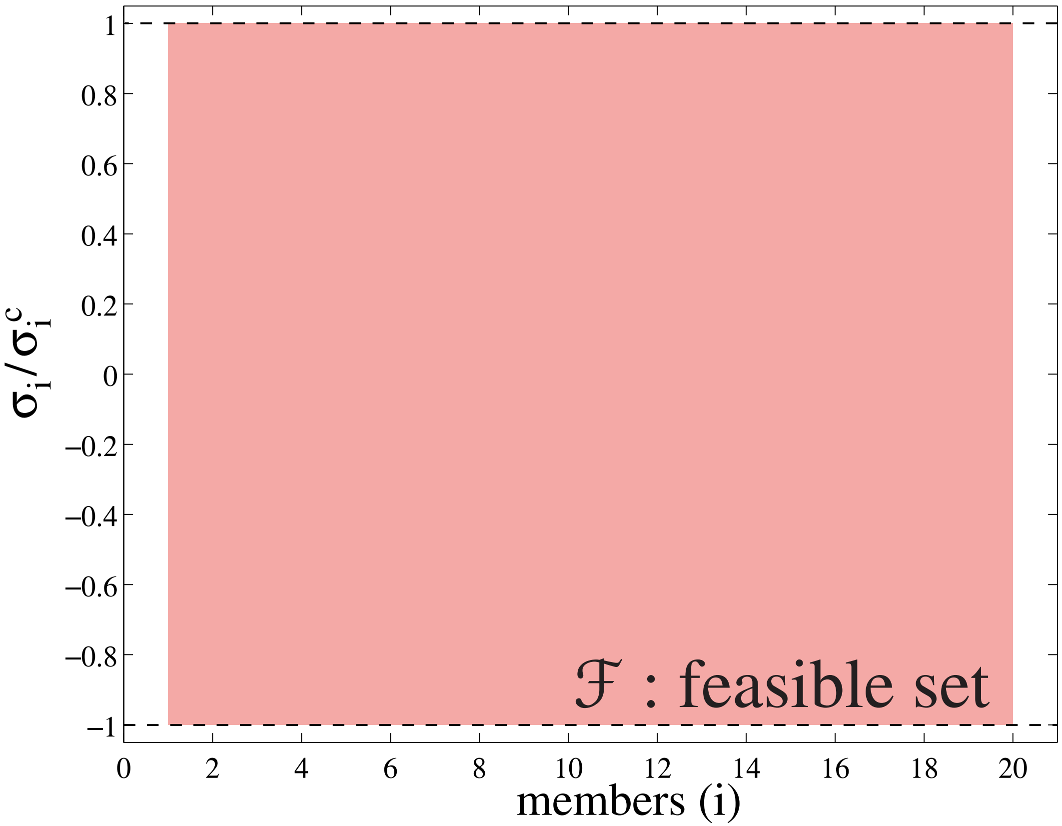
stress states

example (20-bar truss)

- $\mathbf{f} = \widetilde{\boldsymbol{f}} + f^0 \zeta_{\mathrm{f}}$, $\mathbf{a} = \widetilde{\boldsymbol{a}} + \boldsymbol{A}^0 \zeta_{\mathrm{a}}$
- $\alpha \geq \|\zeta_{\mathrm{f}}\|_2$, $\alpha \geq \|\zeta_{\mathrm{a}}\|_\infty$



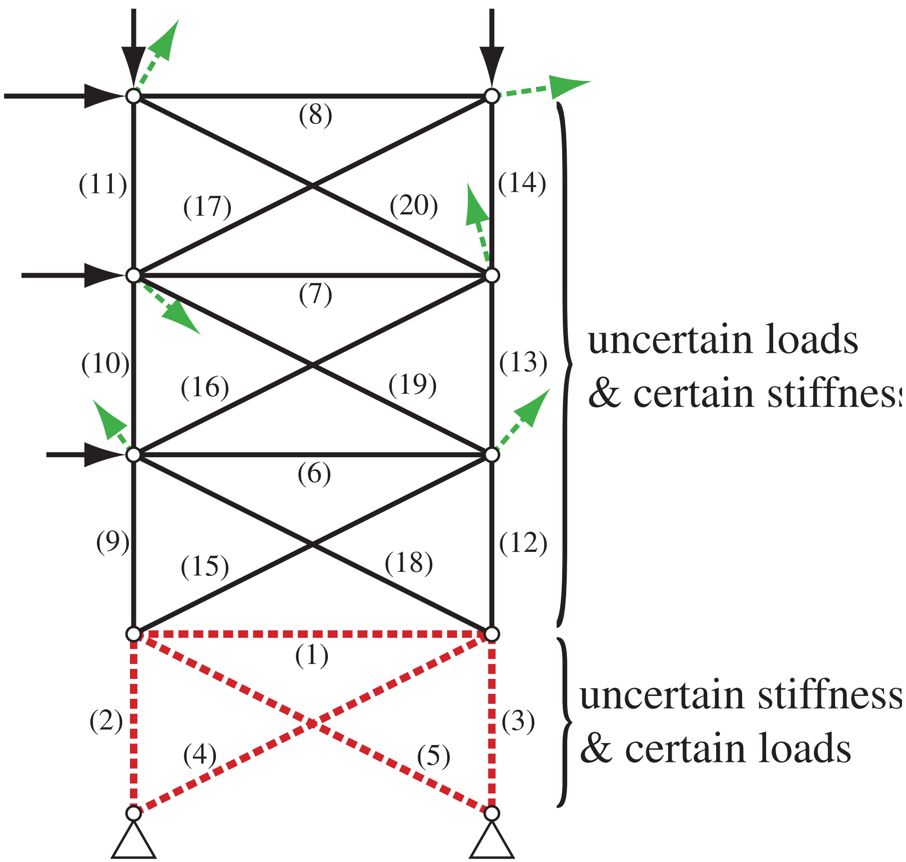
$\widehat{\alpha} = 1.2679$



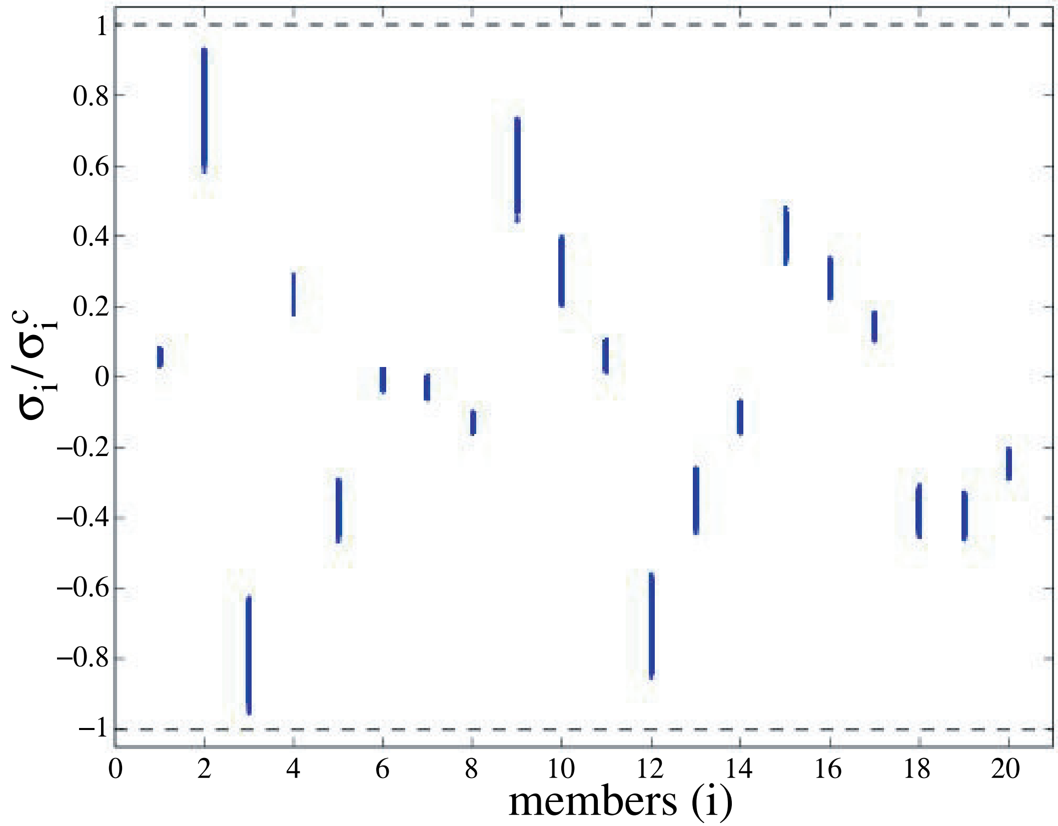
stress states

example (20-bar truss)

- $\mathbf{f} = \widetilde{\boldsymbol{f}} + f^0 \zeta_{\mathrm{f}}$, $\mathbf{a} = \widetilde{\boldsymbol{a}} + \boldsymbol{A}^0 \zeta_{\mathrm{a}}$
- $\alpha \geq \|\zeta_{\mathrm{f}}\|_2$, $\alpha \geq \|\zeta_{\mathrm{a}}\|_\infty$



$\widehat{\alpha} = 1.2679$



stress states

In what follows, we assume that only f has uncertainty

maximization of robustness function $\widehat{\alpha}$

- $\widehat{\alpha}$ depends on $\boldsymbol{a}$ (cross-sectional areas)

$$\widehat{\alpha}(\boldsymbol{a})^2 = \max_{t,\boldsymbol{\rho}}\{t : \boldsymbol{G}(\boldsymbol{a}, t, \boldsymbol{\rho}) \succeq \boldsymbol{O},\ \boldsymbol{\rho} \geq \mathbf{0}\}$$

MAX-$\widehat{\alpha}(\boldsymbol{a})$

$$\max_{\boldsymbol{a}}\left\{\widehat{\alpha}(\boldsymbol{a}) : \boldsymbol{a} \geq \mathbf{0},\ V(\boldsymbol{a}) \leq \overline{V}\right\}$$

maximization of robustness function $\widehat{\alpha}$

- $\widehat{\alpha}$ depends on $\boldsymbol{a}$ (cross-sectional areas)

$$\widehat{\alpha}(\boldsymbol{a})^2 = \max_{t,\boldsymbol{\rho}}\{t : \boldsymbol{G}(\boldsymbol{a}, t, \boldsymbol{\rho}) \succeq \boldsymbol{O},\ \ \boldsymbol{\rho} \geq \mathbf{0}\}$$

MAX-$\widehat{\alpha}(\boldsymbol{a})$

$$\max_{\boldsymbol{a}}\left\{\widehat{\alpha}(\boldsymbol{a}) : \boldsymbol{a} \geq \mathbf{0},\ \ V(\boldsymbol{a}) \leq \overline{V}\right\}$$

nonlinear SDP formulation

$$\max_{\boldsymbol{a},t,\boldsymbol{\rho}}\left\{t\ \ :\ \ \boldsymbol{G}(\boldsymbol{a}, t, \boldsymbol{\rho}) \succeq \boldsymbol{O},\ \ \boldsymbol{\rho} \geq \mathbf{0},\ \ \boldsymbol{a} \geq \mathbf{0},\ \ V(\boldsymbol{a}) \leq \overline{V}\right\} \quad \text{(NL-SDP)}$$

sequential SDP method

NL-SDP

$$\max_{\boldsymbol{x}} \quad \boldsymbol{c}^\top \boldsymbol{x}$$
$$\text{s.t.} \quad \boldsymbol{G}(\boldsymbol{x}) \succeq \boldsymbol{O}$$

- G : nonlinear function $\in \mathcal{S}^{n+1}$
  (G is affine $\Longleftrightarrow$ SDP)

SDP approximation of NL-SDP at x^k

$$\max_{\Delta \boldsymbol{x}} \quad \boldsymbol{c}^\top \Delta \boldsymbol{x} - \frac{1}{2} c^k \|\Delta \boldsymbol{x}\|^2 \qquad (\clubsuit)$$
$$\text{s.t.} \quad \boldsymbol{G}(\boldsymbol{x}^k) + D\boldsymbol{G}^k \cdot \Delta \boldsymbol{x} \succeq \boldsymbol{O}$$

S-SDP for MAX-$\widehat{\alpha}(\boldsymbol{a})$

Step 0: Choose an initial solution $\boldsymbol{a}^0$, and set $k := 0$.

Step 1: Fixing $\boldsymbol{a} = \boldsymbol{a}^k$ in NL-SDP, find an optimum $(t^k, \boldsymbol{\rho}^k)$.

Step 2: Find the (unique) optimal solution $(\Delta t^k, \Delta\boldsymbol{\rho}^k, \Delta\boldsymbol{a}^k)$ of the SDP model (♣).
If $\|(\Delta t^k, \Delta\boldsymbol{\rho}^k, \Delta\boldsymbol{a}^k)\| \leq \epsilon$, then STOP.

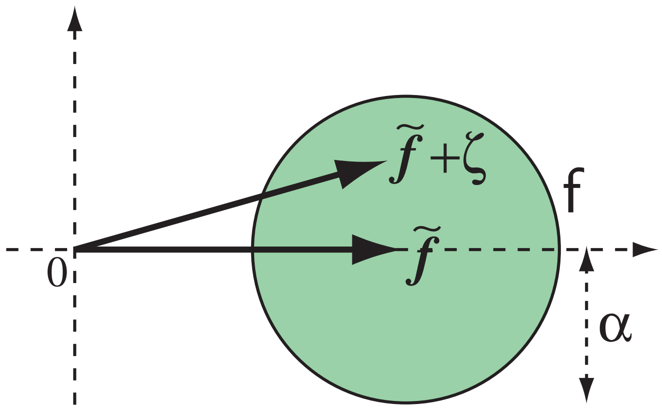
Step 3: Update $\boldsymbol{a}^{k+1} := \boldsymbol{a}^k + \Delta\boldsymbol{a}^k$.

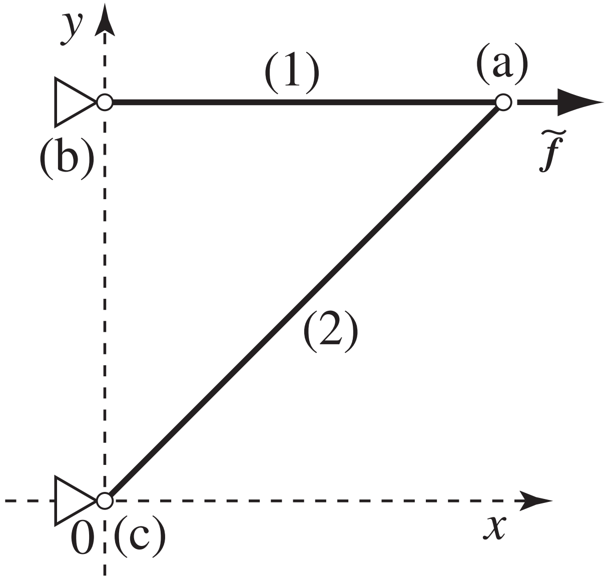
Step 4: Choose $c^{k+1} > 0$. Set $k \leftarrow k + 1$ and go to Step 1.

- global convergence
- SDP models (♣)
  - can be solved by using the Interior-Point Method

MAX-$\widehat{\alpha}(a)$: 2-bar truss

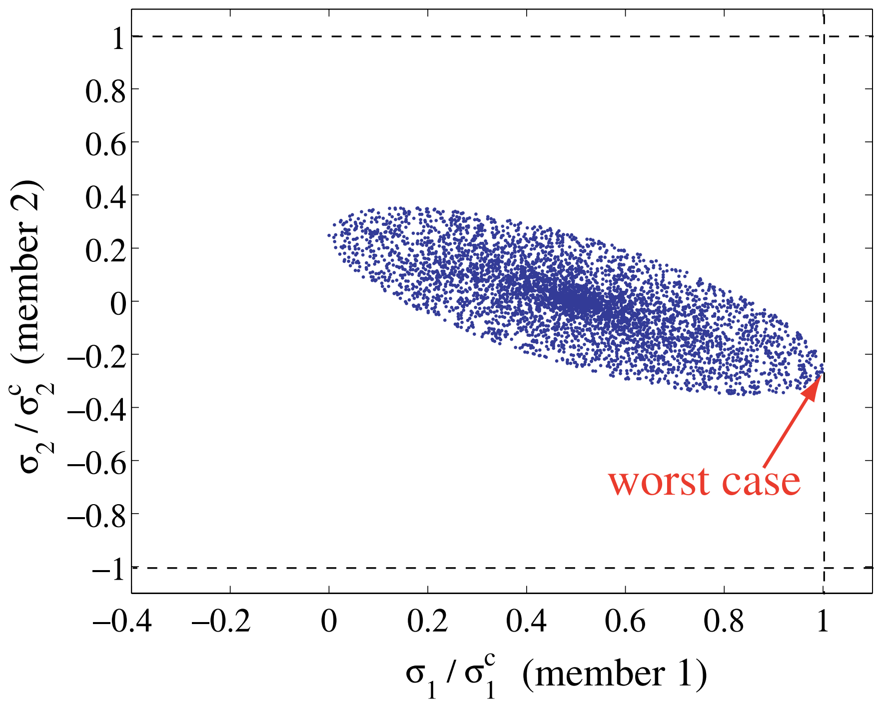
- $\mathsf{f} = \widetilde{f} + \zeta, \quad \alpha \geq \|\zeta\|_2$
- stress constraints
- interior-point method
  - SeDuMi 1.05 [Sturm 99] / Matlab 6.5.1

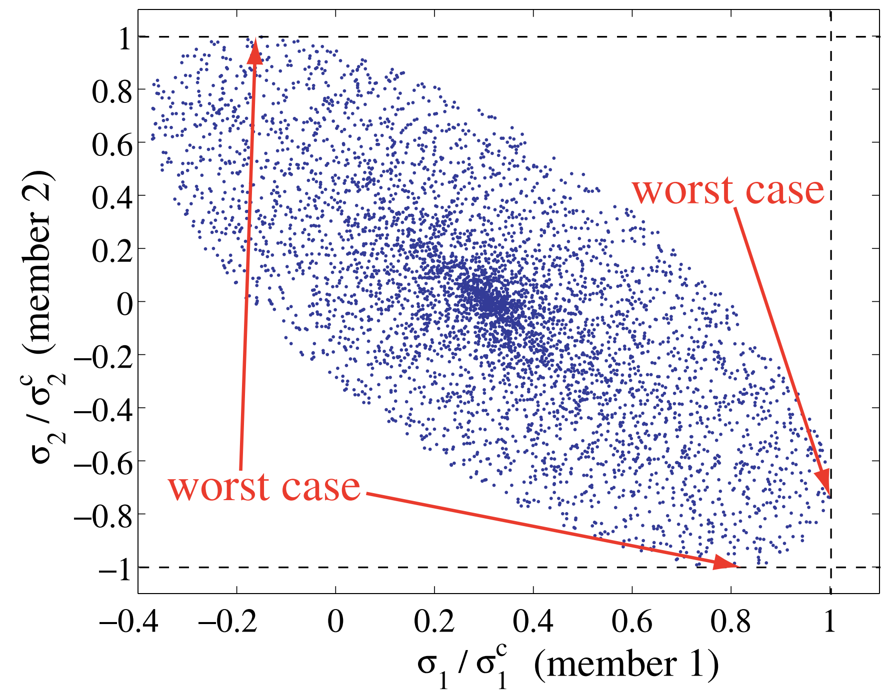




MAX-$\widehat{\alpha}(\boldsymbol{a})$: 2-bar truss

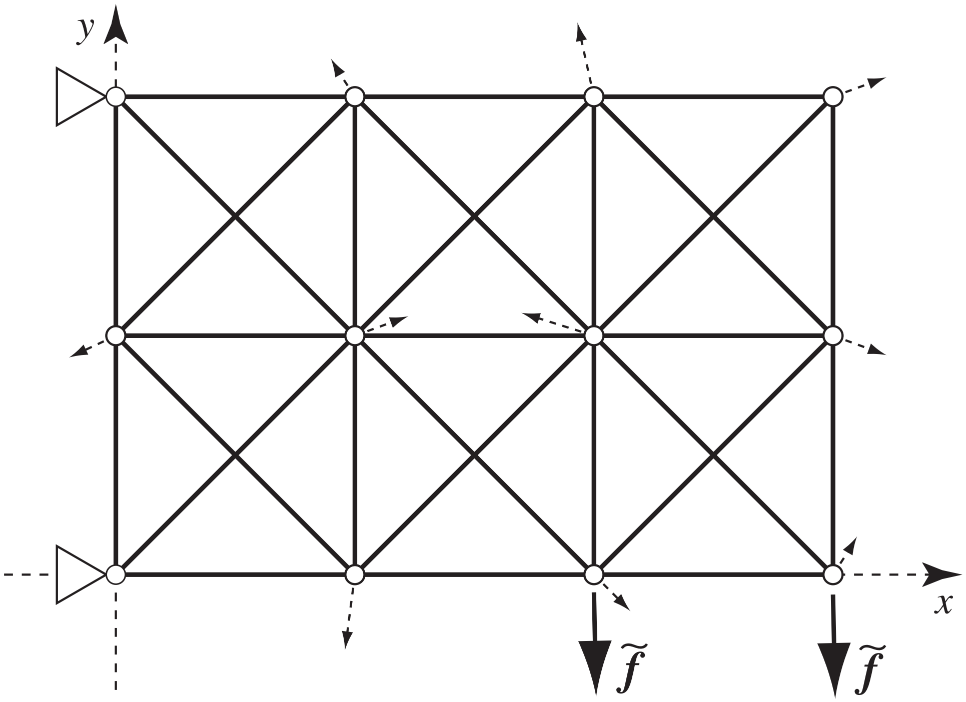
- stresses for randomly generated ζ
- at the optimal design
  $\longrightarrow$ both members encounter worst cases

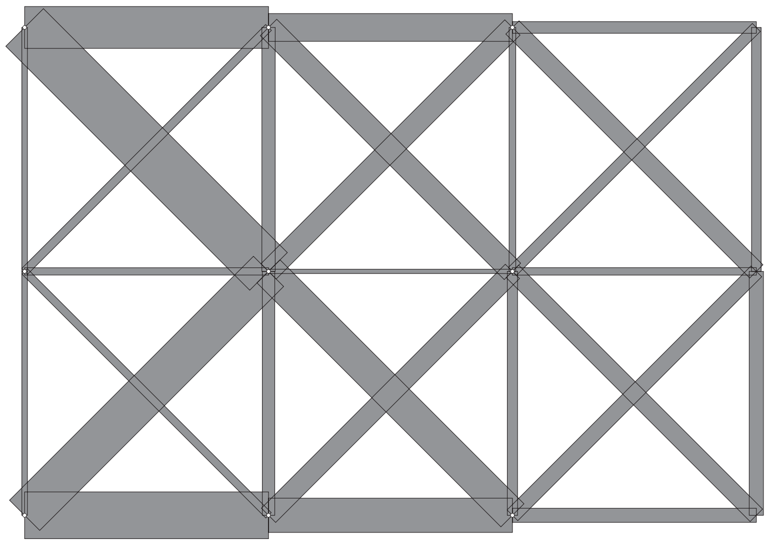




$\widehat{\alpha}(\boldsymbol{a}^0) = 69.3 \text{ kN}$ (initial sol.) $\widehat{\alpha}(\boldsymbol{a}^*) = 153.8 \text{ kN}$ (optimal sol.)

MAX-$\widehat{\alpha}(\boldsymbol{a})$: 29-bar truss

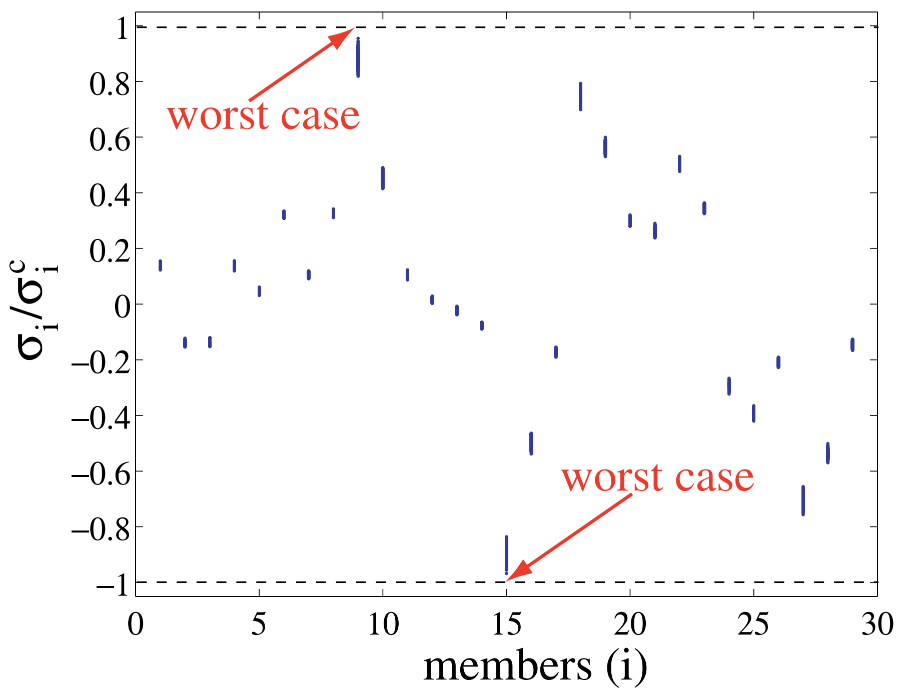


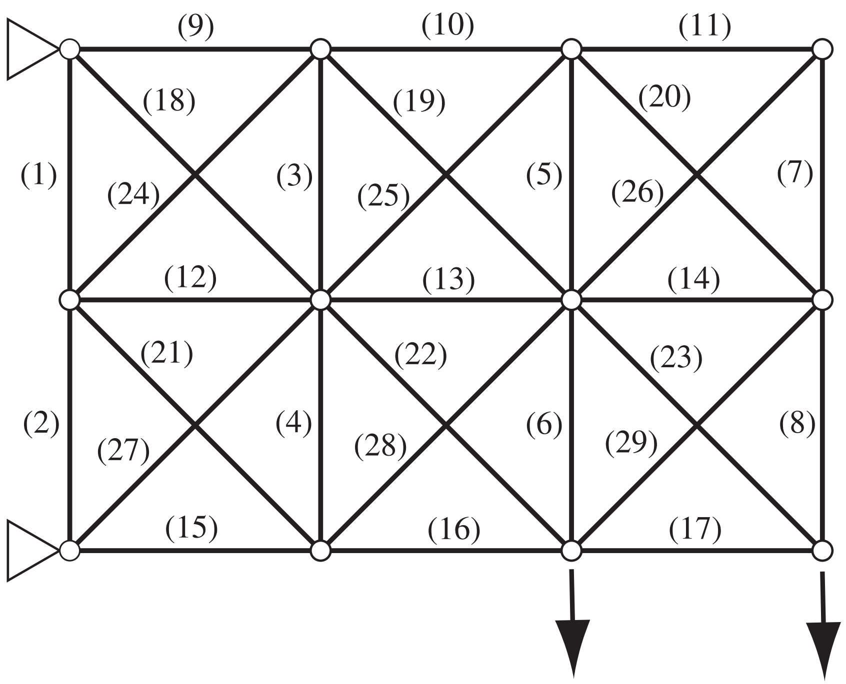


optimal design $\boldsymbol{a}^*$

- uncertain f applied at all nodes
- stress constraints $|\sigma_i| \leq \sigma_i^c$

MAX-$\widehat{\alpha}(a)$: 29-bar truss

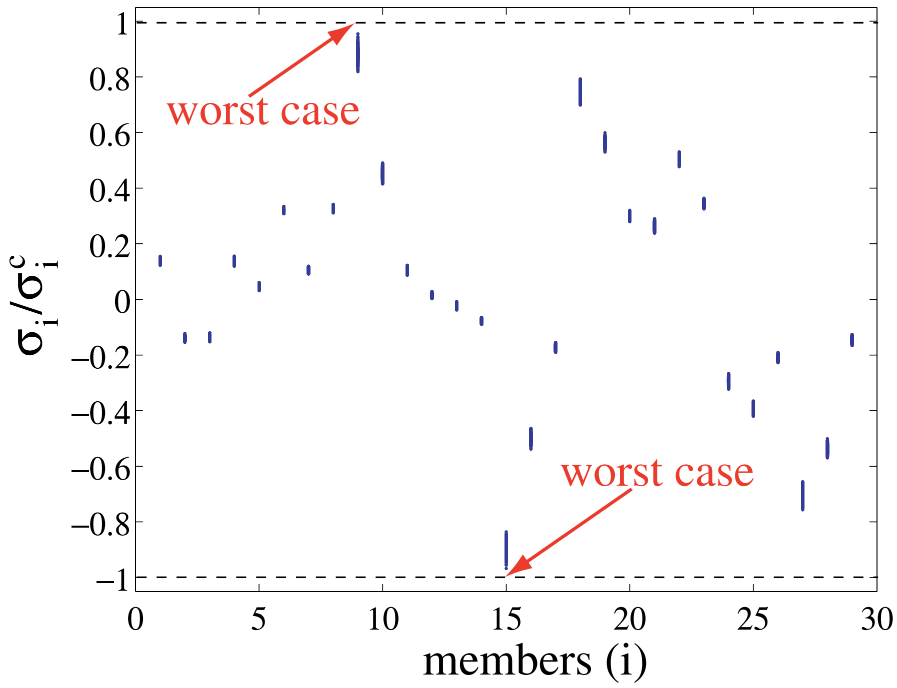




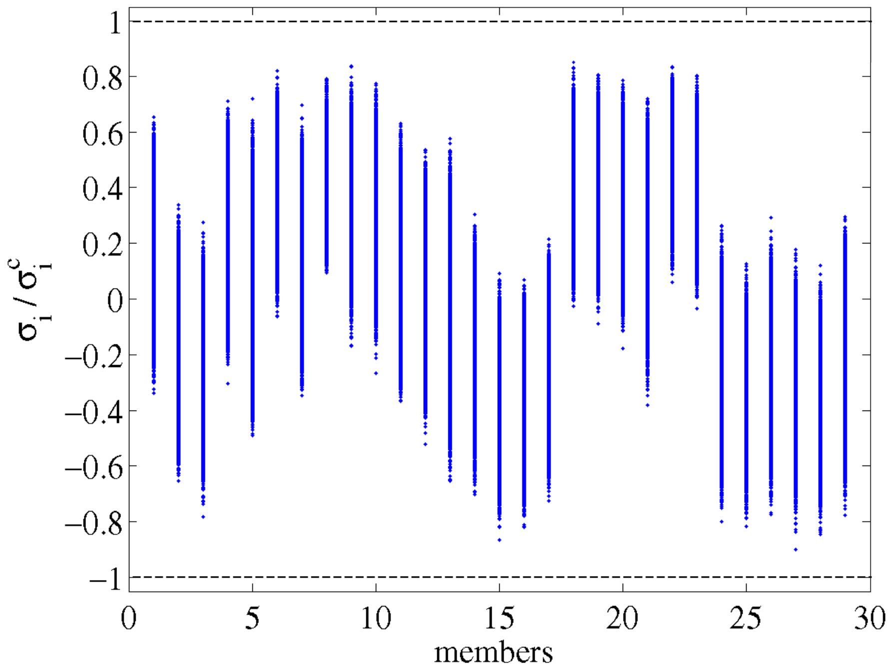
initial sol. a^0

- uncertain f applied at all node
- $\widehat{\alpha}(\boldsymbol{a}^0) = 0.72$ kN
- $\widehat{\alpha}(\boldsymbol{a}^*) = 10.85$ kN

MAX-$\widehat{\alpha}(a)$: 29-bar truss



initial sol. a^0



optimal sol. a^*

- uncertain f applied at all node
- $\widehat{\alpha}(\boldsymbol{a}^0) = 0.72$ kN
- $\widehat{\alpha}(\boldsymbol{a}^*) = 10.85$ kN

conclusions

- robustness function $\widehat{\alpha}(\boldsymbol{a})$
  - measure of robustness
  - uncertain loads / stiffness
  - quadratic embedding $+\ \mathcal{S}$-procedure
  - can be found by solving an SDP
- MAX-$\widehat{\alpha}(\boldsymbol{a})$
  - nonlinear SDP
  - successive SDP method
  - find the optimal design by solving SDPs successively