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# Generating Asymmetric Tensegrity Structures via Truss Topology Optimization

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**Summary:** This paper presents a truss topology optimization approach to generating many different tensegrity structures from a given initial solution. Symmetry of the solutions is implicitly controlled by using a constraint on the number of different member lengths. The topology optimization problem is solved with the mixed-integer linear programming.

## 1. Introduction

A *tensegrity structure* is a free-standing prestressed pin-jointed structure consisting of a set of discontinuous compressive components (struts) interacting with a set of continuous tensile components (cables). Many classical tensegrity structures in literature are kinematically indeterminate; see, e.g., Calladine [1] and Hanaor and Liao [3]. Those tensegrity structures, therefore, have infinitesimal mechanism(s). However, they are stabilized by introducing prestress forces. Such a structure is said to be *prestress stable* [2].

This paper presents a numerical method for generating many different tensegrity structures from one initial solution. The method is based on truss topology optimization. Particular attention is given to variety of symmetry properties in configurations of generated tensegrity structures.

One of most difficult constraints in designing tensegrity structures is the discontinuity condition of struts. Therefore, many existing form-finding methods require specifying topology, i.e., connectivity relation of cables and struts, of a tensegrity structure as input data; see, e.g., a survey due to Juan and Mirats Tur [5]. For exploring new tensegrity structures, however, specifying topology may be too restrictive. As a method requiring no information of topology in advance, the author proposed a mixed integer programming approach to topology optimization of tensegrity structures [6, 7]. However, almost all tensegrity structures obtained by this method are kinematically determinate, i.e., stable. In this paper we thus use a constraint ensuring simultaneous static and kinematic indeterminacy [8], that is formulated based upon the extended Maxwell's counting rule for rigidity [1, 11]. Numerical experiments illustrate that prestress stable tensegrity structures are often obtained by minimizing the total length of cables.

Many studies have been made on finding tensegrity structures with high symmetry in configurations. In contrast, recent interest has been drawn for developing versatile numerical methods that can generate diverse non-symmetric tensegrity structures. This paper proposes to make use of a constraint on the number of different member lengths to achieve diversity of symmetry properties of solutions. As this number increases, symmetry of a structure becomes lower. Therefore, by controlling this parameter we can generate tensegrity structures with various symmetry properties from one given initial structure. In other words, symmetry of solutions is implicitly controlled by this parameter. For instance, if the number of different member lengths is set equal to the number of existing members, then it is guaranteed that a tensegrity structure without symmetry can be obtained.

Concerning symmetry property, in this paper we use two parameters, the number of different strut lengths and the number of different cable lengths. Also, the lower bound for the number of struts is used as the third parameter. The truss topology optimization problem with this constraint, together with the other constraints expressing the definition of tensegrity structure [7], is formulated as a *mixed-integer linear programming* (MILP) problem. Several well-developed software packages, e.g., CPLEX [4], are available to solve an MILP problem globally. Unlike the conventional form-finding methods, the locations of nodes are fixed throughout the proposed method.

## 2. Integer variables for member labels

We use the conventional ground structure method for truss topology optimization to generate tensegrity structures. An initial structure consists of sufficiently many candidate members. Locations of the nodes are specified. We use $V$ and $E$ to denote the set of nodes and the set of candidate members, respectively.

We remove some members from the initial structure to obtain a tensegrity structure. Let $S$, $C$, and $N$ denote the sets of struts, cables, and removed members, respectively. Topology of a tensegrity structure is determined by finding a partition of $E$ into disjoint subsets $E = S \cup C \cup N$. A key idea proposed in Kanno [7, 8] is making use of integer variables that serve as labels of members. Specifically, we use two 0–1 variables, $x_i$ and $y_i$, to express the label of member $i$ as

$$(x_i, y_i) = (1,0) \quad \Leftrightarrow \quad i \in S, \tag{1a}$$
$$(x_i, y_i) = (0,1) \quad \Leftrightarrow \quad i \in C, \tag{1b}$$
$$(x_i, y_i) = (0,0) \quad \Leftrightarrow \quad i \in N. \tag{1c}$$

A classical definition of tensegrity structure requires that each node is connected to at most one strut [10]. This condition, called the *discontinuity condition of struts*, can be expressed in terms of $x_i$ in (1) as

$$\sum_{i \in E(v_p)} x_i \le 1, \quad \forall v_p \in V. \tag{2}$$

Here, $E(v_p) \subseteq E$ is the set of indices of the members that are connected to node $v_p \in V$. Besides, various constraints that are indispensable in design problem of tensegrity structures can be formulated in terms of $x_i$ and $y_i$ [7, 8]. Particularly, most of classical tensegrity structures in literature are kinematically indeterminate [1, 3]. To address this issue, a constraint on $x_i$ and $y_i$ based upon the modern Maxwell's counting rule for rigidity [1, 11] can be used; see Kanno [8] for details.

## 3. Symmetry constraint

Symmetry of structures often provides both practical and theoretical advantages in applications. It is also related to beauty and simplicity. In particular, many well-known tensegrity structures have symmetric configurations such as (semi-)regular polyhedra. On the other hand, for diversity of tensegrity structures, it is favorable that a design space explored by a design method is not limited to symmetric configurations. Indeed, design methods for finding non-symmetric tensegrity structures have gotten much attention recently [9, 12, 13].

An essential idea for controlling symmetry of a solution is to specify the number of different member lengths of the solution. High symmetry in geometry generally implies that the number of different member lengths is relatively small. In other words, if the number of different member lengths is large, then the geometry of the structure has low, or no, symmetry. For instance, if all members have different lengths, then it is clear that geometry of the structure has no symmetry. Therefore, by specifying a large number of different member lengths as a constraint of the optimization problem, we obtain an optimal structure with low symmetry.

Let $b_{\rm s}$ and $b_{\rm c}$ denote the number of different strut lengths and the number of different cable lengths, respectively. We specify these two parameters in the truss topology optimization problem that we solve for generating a tensegrity structure. These constraints are formulated in terms of $x_i$ and $y_i$ as follows.

Let $l_i$ denote the length of member $i$. We denote by $E_j \subseteq E$ the set of members that have the same member lengths. In other words, $i \in E_j$ and $i' \in E_j$ mean $l_i = l_{i'}$. Suppose that the initial structure has $b$ different member lengths. Then we obtain a partition of $E$ into $b$ disjoint subsets:

$$E = E_1 \cup \cdots \cup E_b.$$

We write $B = \{1, \dots, b\}$ for simplicity. Define 0–1 variables $z_j$ $(j \in B)$ by

$$z_j = \begin{cases} 0 & \text{if } S \cap E_j = \emptyset, \\ 1 & \text{if } S \cap E_j \neq \emptyset. \end{cases} \tag{3}$$

The number of different strut lengths is specified as

$$\sum_{j \in B} z_j = b_{\rm s}, \tag{4}$$

where $b_{\rm s}$ is a specified value. Constraint (3) can be written by using $x_i$ $(i \in E)$ as

$$z_j \le \sum_{i \in E_j} x_i, \quad z_j \ge \frac{1}{|E_j|} \sum_{i \in E_j} x_i, \quad z_j \in \{0, 1\}, \tag{5}$$

where $|E_j|$ is the number of members belonging to $E_j$. Thus the constraint on the number of different strut lengths can be expressed by (4) and (5), which are handled within the framework of mixed-integer linear programming.

Similarly, the constraint on the number of different cable lengths can be formulated using additional 0–1 variables. Define $w_j$ $(j \in B)$

$$w_j = \begin{cases} 0 & \text{if } C \cap E_j = \emptyset, \\ 1 & \text{if } C \cap E_j \neq \emptyset. \end{cases}$$

Then the number of different cable lengths is constrained by the following constraints:

$$\sum_{j \in B} w_j = b_{\rm c},$$

$$w_j \le \sum_{i \in E_j} y_i, \quad w_j \ge \frac{1}{|E_j|} \sum_{i \in E_j} y_i, \quad w_j \in \{0, 1\} \qquad (\forall j \in B),$$

where $b_{\rm c}$ is a specified value.

As for the objective function of the optimization problem, we attempt to minimize the total length of cables. The conditions required by the definition of tensegrity structure, including kinematical indeterminacy, are considered as constraints. Then the optimization problem can be formulated as a mixed-integer linear programming problem; see Kanno [8] for more accounts.

The global optimal solution of a mixed-integer linear programming problem can be found by using, e.g., a branch-and-cut algorithm. Several well-developed software packages, e.g., CPLEX [4], are available for solving this optimization problem.

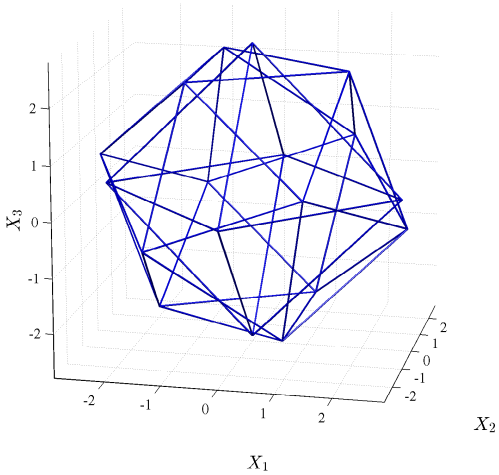


Fig. 1: An 18-node initial structure.

**4. Examples**

Various tensegrity structures are found by solving the proposed MILP problem with controlling three parameters, i.e., the lower bound for the number of struts, $s$, the number of different strut lengths, $b_{\rm s}$, and the number of different cable lengths, $b_{\rm c}$. The MILP problem was solved with CPLEX ver. 12.2 [4]. The tolerance of integrality feasibility of the solver was set equal to $10^{-8}$. The other parameters were set as the default values. The bounds for prestress forces are given as $-200\,\mathrm{kN} \le q_i \le -20\,\mathrm{kN}$ for struts and $10\,\mathrm{kN} \le q_i \le 150\,\mathrm{kN}$ for cables. To ensure kinematical indeterminacy of tensegrity structures, we used the constraint $d_{\rm k} - d_{\rm s} = 1$ [7], where $d_{\rm k}$ is the degree of kinematical indeterminacy and $d_{\rm s}$ is the degree of statical indeterminacy. Accordingly, all the tensegrity structures obtained in this section are prestress stable.

Table 3: Member lengths of tensegrity structures in Figure 4.

| | Figure 4(a) | | Figure 4(b) | |
|---|---|---|---|---|
| | $l_i$ (m) | | $l_i$ (m) | |
| strut | 3.00000 | ×2 | 4.85410 | ×4 |
| | 4.85083 | ×2 | 4.97832 | ×2 |
| | 4.85410 | ×2 | 5.26060 | ×2 |
| | 5.26060 | ×2 | 5.40000 | ×1 |
| cable | 1.52463 | ×4 | 0.54496 | ×2 |
| | 1.78529 | ×6 | 1.52463 | ×4 |
| | 2.70750 | ×4 | 3.00000 | ×22 |
| | 3.00000 | ×18 | 3.54123 | ×4 |
| | 5.40000 | ×1 | 3.81838 | ×6 |

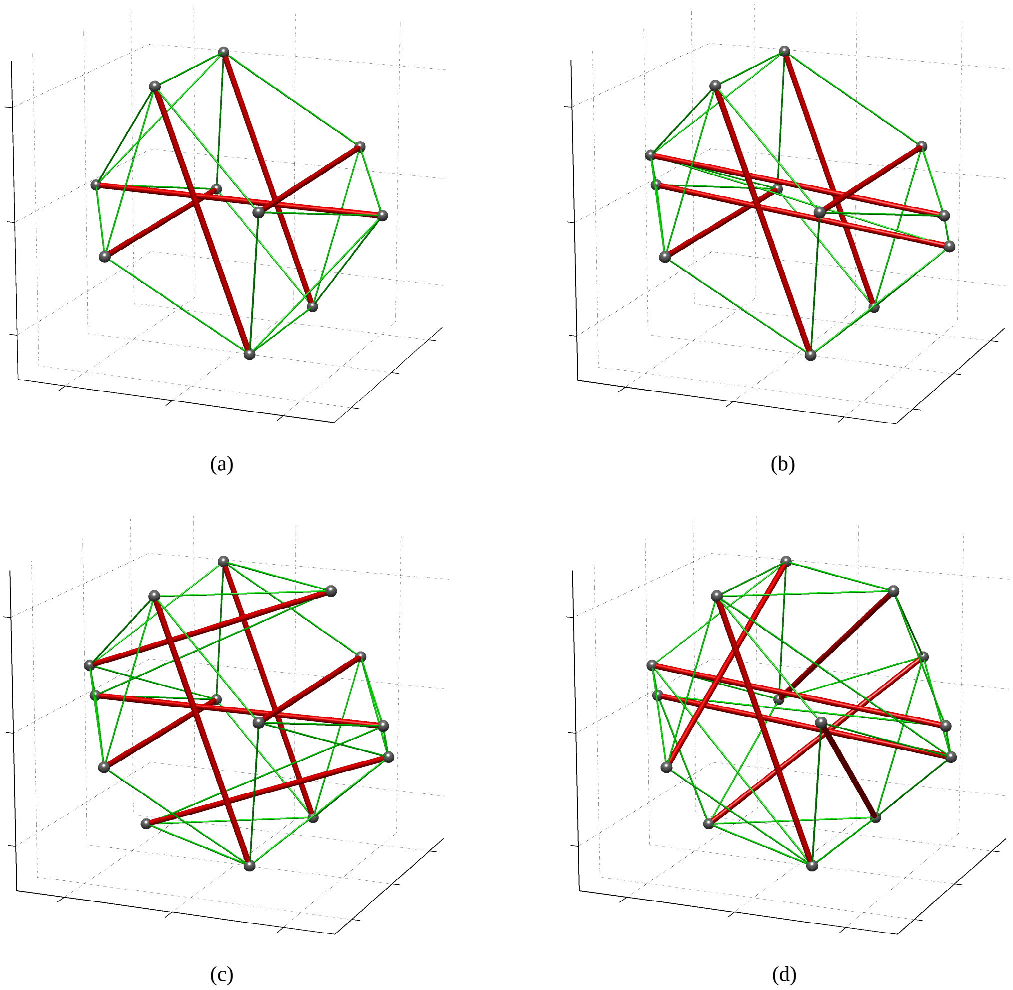


Fig. 2: Tensegrity structures with $b_\mathrm{s} = 2$ different strut lengths. (a) $(s, b_\mathrm{s}, b_\mathrm{c}) = (5, 2, 3)$; (b) $(s, b_\mathrm{s}, b_\mathrm{c}) = (6, 2, 4)$; (c) $(s, b_\mathrm{s}, b_\mathrm{c}) = (7, 2, 4)$; and (d) $(s, b_\mathrm{s}, b_\mathrm{c}) = (7, 2, 5)$.

Table 1: Member lengths of tensegrity structures in Figure 2.

| | Figure 2(a) | | Figure 2(b) | | Figure 2(c) | | Figure 2(d) | |
|---|---|---|---|---|---|---|---|---|
| | $l_i$ (m) | | $l_i$ (m) | | $l_i$ (m) | | $l_i$ (m) | |
| strut | 4.85410 | ×4 | 4.85410 | ×4 | 4.85410 | ×6 | 4.85410 | ×5 |
| | 5.40000 | ×1 | 5.52849 | ×2 | 5.40000 | ×1 | 5.52849 | ×2 |
| cable | 2.80424 | ×4 | 0.54496 | ×2 | 0.54496 | ×2 | 0.54496 | ×2 |
| | 3.00000 | ×10 | 2.80424 | ×4 | 2.80424 | ×4 | 2.46526 | ×2 |
| | 3.27934 | ×4 | 3.00000 | ×16 | 3.00000 | ×20 | 3.00000 | ×21 |
| | | | 5.70634 | ×1 | 4.97832 | ×2 | 4.85410 | ×2 |
| | | | | | | | 5.40000 | ×1 |

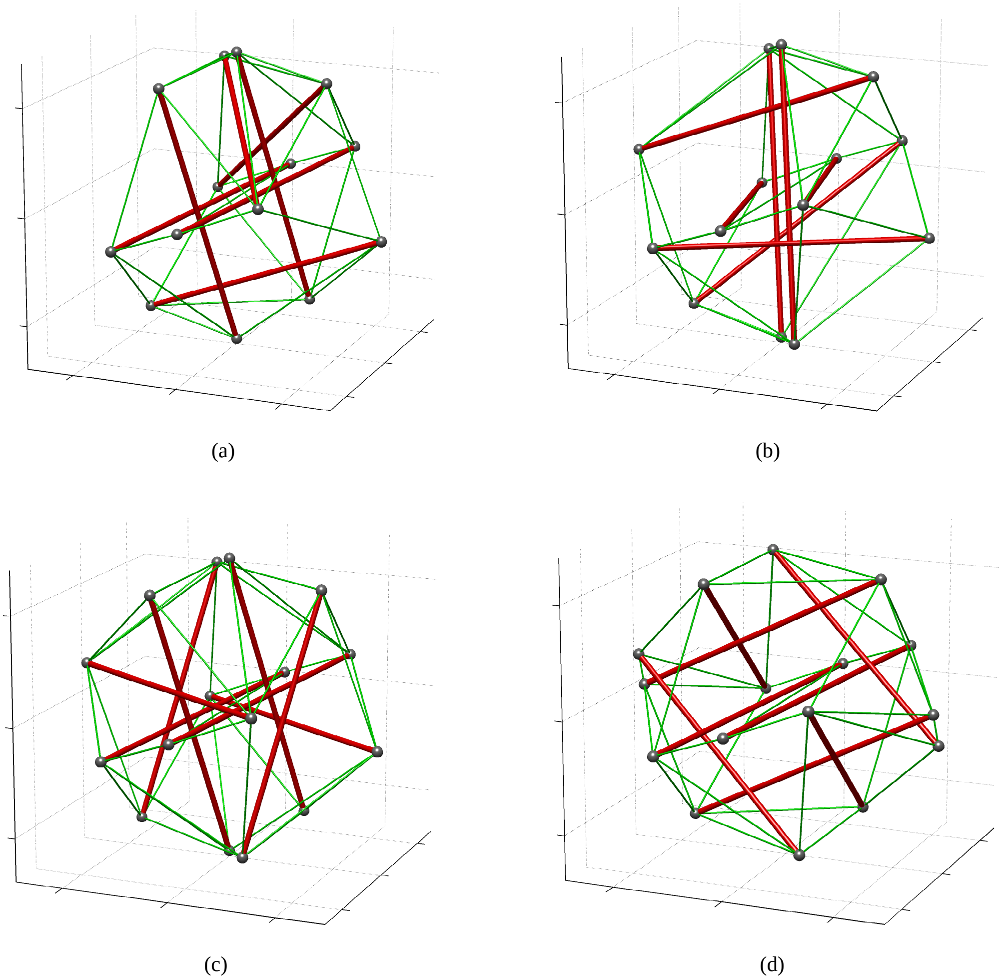


Fig. 3: Tensegrity structures with $b_{\mathrm{s}} = 3$ different strut lengths. (a) $(s, b_{\mathrm{s}}, b_{\mathrm{c}}) = (7,3,5)$; (b) $(s, b_{\mathrm{s}}, b_{\mathrm{c}}) = (7,3,6)$; (c) $(s, b_{\mathrm{s}}, b_{\mathrm{c}}) = (8,3,4)$; and (d) $(s, b_{\mathrm{s}}, b_{\mathrm{c}}) = (8,3,6)$.

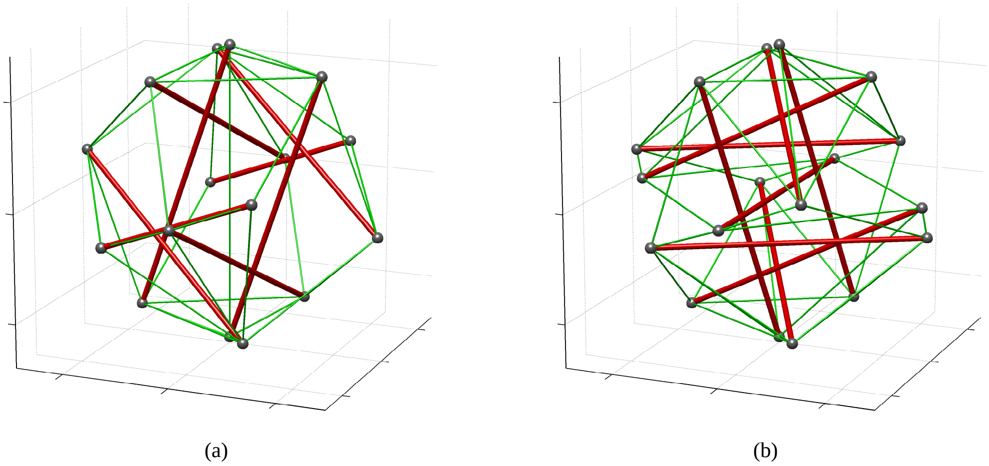


Fig. 4: Symmetric tensegrity structures with $b_{\mathrm{s}} = 4$ different strut lengths. (a) $(s, b_{\mathrm{s}}, b_{\mathrm{c}}) = (8,4,5)$; and (b) $(s, b_{\mathrm{s}}, b_{\mathrm{c}}) = (9,4,5)$.

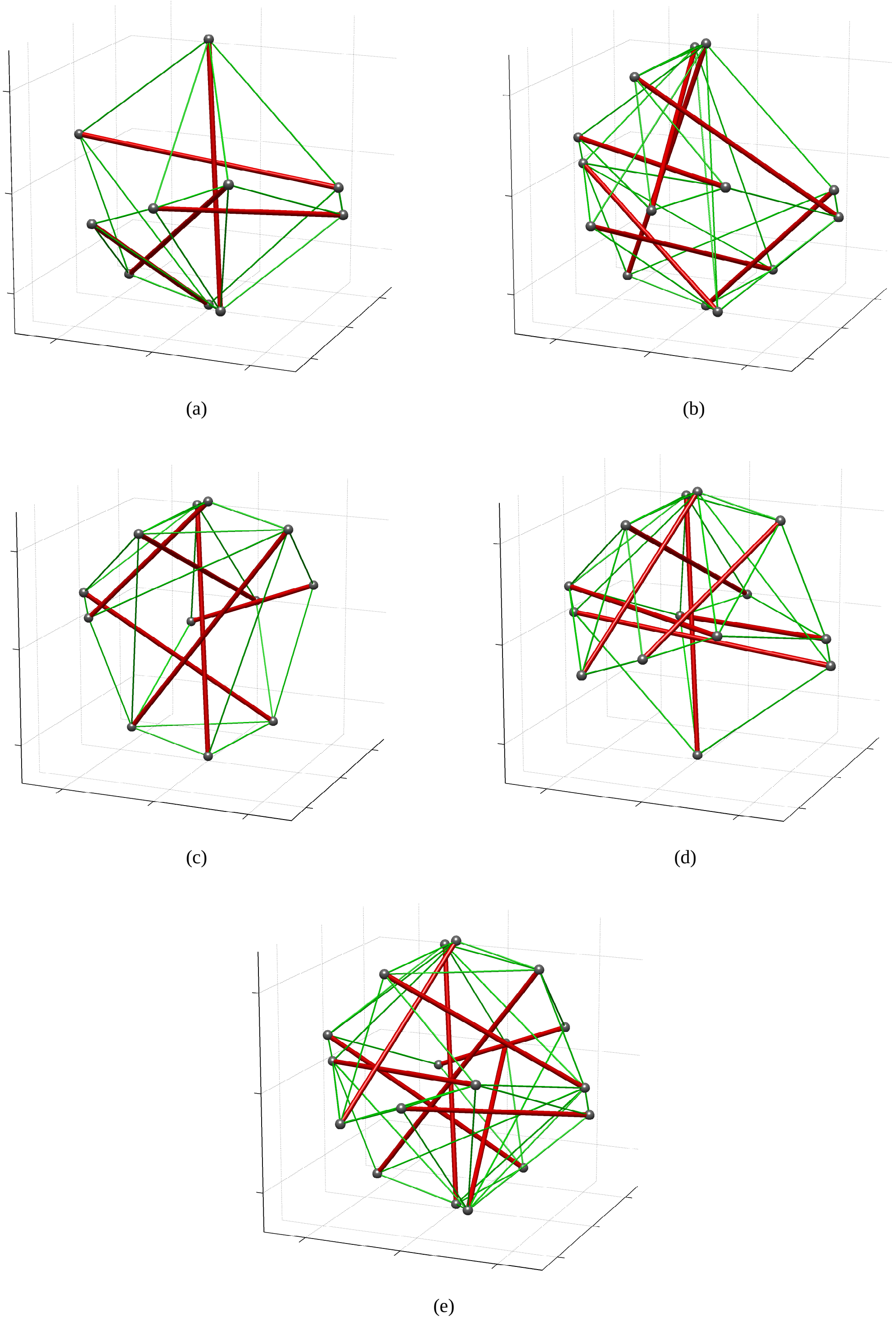


Fig. 5: Tensegrity structures with asymmetric configurations. (a) $(s, b_\mathrm{s}, b_\mathrm{c}) = (5, 5, 8)$; (b) $(s, b_\mathrm{s}, b_\mathrm{c}) = (5, 5, 18)$; (c) $(s, b_\mathrm{s}, b_\mathrm{c}) = (6, 6, 9)$; (d) $(s, b_\mathrm{s}, b_\mathrm{c}) = (7, 7, 10)$; and (e) $(s, b_\mathrm{s}, b_\mathrm{c}) = (9, 9, 13)$.

Figure 1 shows the initial structure. Here, $X_1$ and $X_2$ are taken to be two horizontal axes and $X_3$ is the vertical axis. The structure consists of $|V| = 18$ nodes and $|E| = 153$ members. The locations of the nodes of this initial structure are defined as the vertices of regular polyhedra centered at the origin. The 12 nodes are at vertices of a regular icosahedron with edge length 3 m, while the remaining 6 nodes are at vertices of a regular octahedron with edge length $2.7\sqrt{2}$ m. Any two nodes are connected by a member, although only edges of polyhedra are illustrated in Figure 1. The number of different member lengths of this initial structure is $|B| = 22$. Presence of mutually intersecting members in a tensegrity structure is avoided. More precisely, member $i$ and member $i'$ cannot exist simultaneously if the distance of member $i$ and member $i'$ is less than $\delta = 0.1$ m. This constraint is also handled within the framework of MILP [7].

We explore new tensegrity structures by increasing the number of different strut lengths, $b_{\rm s}$, form a small value. The solutions obtained by solving MILP problems with $b_{\rm s} = 2$ are shown in Figure 2, where the thick lines and the thin lines represent struts and cables, respectively. The number of different cable lengths, $b_{\rm c}$, of each structure is also relatively small. Recall that if the geometry of a structure has high symmetry, then the number of different member lengths is small. Lengths of struts and cables are listed in Table 1. For instance, Figure 2(a) consists of 5 struts with $b_{\rm s} = 2$ different lengths and 18 cables with $b_{\rm c} = 3$ different lengths. Thus this tensegrity structure involves only 5 different member lengths. Similarly, the tensegrity structure in Figure 2(c) involves only $b_{\rm s} + b_{\rm c} = 2 + 4 = 6$ different member lengths, while it has 7 struts and 28 cables. Accordingly, tensegrity structures shown in Figures 2(a), (b), and (c) have symmetric configurations. In contrast, the one shown in Figure 2(d) is not symmetric. Thus the small number of different member lengths does not necessarily imply high symmetry in configuration.

Figure 3 and Table 2 collect the tensegrity structures obtained by solving MILP problems with $b_{\rm s} = 3$. Among these tensegrity structures, the ones shown in Figures 3(a) and (b) have asymmetric configurations. In contrast, the tensegrity structures in Figures 3(c) and (d) have symmetric configurations. Figure 4 and Table 3 also collect symmetric tensegrity structures. These tensegrity structures have $b_{\rm s} = 4$ different strut lengths.

Figure 5 shows asymmetric tensegrity structures. All struts of each structure have different lengths. It is observed that the large number of different member lengths implies low, or no, symmetry in configuration. Thus new asymmetric tensegrity structures can be found easily by controlling parameters $b_{\rm s}$ and $b_{\rm c}$ in the proposed method.

### 5. Conclusion

For exploring innovative use of tensegrity structures in real applications, systematic approaches that can generate diverse tensegrity structures are desired. This paper has presented an optimization-based method for generating various topologies of tensegrity structures that satisfy the classical definition rigorously. A mixed-integer linear programming has been used for solving the optimization problem.

It has been shown through the numerical experiments that the proposed method can often find prestress stable tensegrity structures. Moreover, it can generate many different tensegrity structures from one initial structure by controlling only three parameters, i.e., the lower bound for the number of struts, the number of different strut lengths, and the number of different cable lengths. Also, by controlling the latter two parameters, symmetry in the configuration of a tensegrity structure is implicitly controlled. It has been illustrated in the numerical examples that the proposed method can generate both symmetric and asymmetric tensegrity structures from one initial symmetric structure.

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Table 2: Member lengths of tensegrity structures in Figure 3.

| | Figure 3(a) | | Figure 3(b) | | Figure 3(c) | | Figure 3(d) | |
|---|---|---|---|---|---|---|---|---|
| | $l_i$ (m) | | $l_i$ (m) | | $l_i$ (m) | | $l_i$ (m) | |
| strut | 4.85410 | ×3 | 4.85410 | ×3 | 4.85410 | ×4 | 4.85410 | ×4 |
| | 5.26060 | ×2 | 5.26060 | ×2 | 5.26060 | ×2 | 4.97832 | ×2 |
| | 5.34197 | ×2 | 5.34197 | ×2 | 5.34197 | ×2 | 5.34197 | ×2 |
| cable | 1.52463 | ×4 | 1.52463 | ×4 | 1.52463 | ×4 | 0.54496 | ×2 |
| | 1.78529 | ×4 | 1.78529 | ×4 | 3.00000 | ×22 | 1.52463 | ×4 |
| | 3.00000 | ×15 | 3.00000 | ×16 | 3.54123 | ×6 | 2.46526 | ×2 |
| | 3.54123 | ×4 | 3.54123 | ×2 | 5.40000 | ×1 | 2.80424 | ×4 |
| | 5.40000 | ×1 | 4.28028 | ×1 | | | 3.00000 | ×20 |
| | | | 5.40000 | ×1 | | | 5.40000 | ×1 |